NATIONAL TECHNICAL UNIVERSITY OF ATHENS
SCHOOL OF CIVIL ENGINEERING
Institute of Structural Analysis & Aseismic Research
Zografou Campus, GR.-157 73 Athens, GREECE

ΔΟΜΟΣΤΑΤΙΚΟΣ ΣΧΕΔΙΑΣΜΟΣ
ΚΑΙ ΑΝΑΛΥΣΗ ΤΩΝ ΚΑΤΑΣΚΕΥΩΝ
Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
Εθνικό Μετσόβιο Πολυτεχνείο

and

ADERS
MSc in Analysis and Design of Earthquake Resistant Structures

Advanced Dynamics of Structures

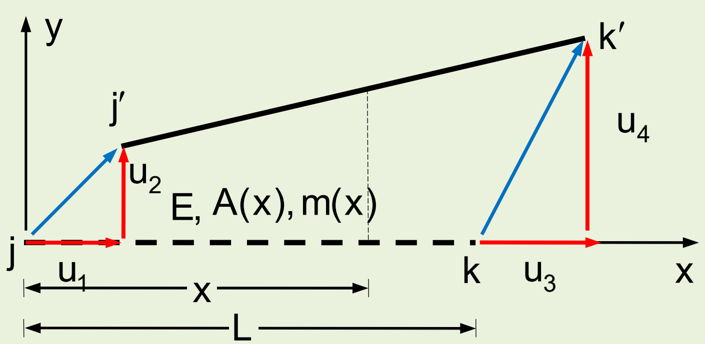
by J.T. Katsikadelis

Lecture # 7

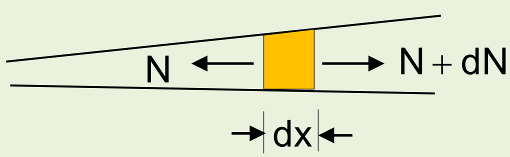
Chapter 11

The Finite Element Method (FEM)

1. THE FEM FOR THE PLANE TRUSS



Degrees of freedom of truss element



Degrees of freedom $u_1(t)$, $u_2(t)$ $u_3(t)$, $u_4(t)$ nodal coordinates

The axial deformation is assumed of the form

$$u(x,t) = u_1(t)\psi_1(x) + u_3(t)\psi_3(x) \qquad (1)$$

where $\psi_1(x)$, $\psi_3(x)$ shape functions

$$\frac{dN}{dx} = 0 \qquad (2)$$

$$N = A(x)\sigma_x = EA(x)\varepsilon_x = EA(x)\frac{d\psi_i}{dx}, \quad i = 1,2 \qquad (3)$$

$$\frac{d}{dx}\left[EA(x)\frac{d\psi_i}{dx}\right] = 0 \qquad (4)$$

Integration gives

$$\psi_i = c_1\int\frac{dx}{A(x)} + c_2\,,\ c_1, c_2 \text{ arbitrary constants} \qquad (5)$$

Constant cross section

$$\psi_i = c_1 x + c_2 \qquad (6)$$

For $\psi_1(0)=1$ and $\psi_1(L)=0$ gives $\psi_1(x)=1-\xi,\quad \xi = x/L$

For $\psi_3(0)=0$ and $\psi_3(L)=1$ gives $\psi_3(x)=\xi,\quad \xi = x/L$

The transverse deformation is rigid body motion

$$v(x,t) = u_2(1-\xi) + u_4\xi = u_2\psi_2(x) + u_4\psi_4(x) \qquad (7)$$

Note that $\psi_2(x)=\psi_1(x),\quad \psi_4(x)=\psi_3(x)$

THE EQUIVALENT NODAL FORCES

Use of Langrage equations with $A=0$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

Elastic nodal forces $f_{Si} = \dfrac{\partial U}{\partial q_i}$, Inertia nodal forces $f_{Ii} = \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i}$, External nodal forces $p_i(t) = Q_i$

ELEMENT ELASTIC ENERGY AND STIFFNESS MATRIX

$$U = \frac{1}{2}\int_V \sigma_x \varepsilon_x dV$$

$$\sigma_x = E\varepsilon_x,\ \varepsilon_x = \partial u / \partial x$$

$$U = \frac{1}{2}\int_0^L \int_A E\left(\frac{\partial u}{\partial x}\right)^2 dxdydz$$

$$= \frac{1}{2}\int_0^L EA(x)\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$U(u_1, u_3) = \frac{1}{2}\int_0^L EA(x)\left[u_1\psi_1'(x) + u_3\psi_3'(x)\right]^2 dx$$

For, $i = 1,2,3,4$

$$f_{S1} = \frac{\partial U}{\partial u_1} = k_{11}u_1 + k_{13}u_3$$

$$f_{S2} = \frac{\partial U}{\partial u_2} = 0$$

$$f_{S3} = \frac{\partial U}{\partial u_3} = k_{31}u_1 + k_{33}u_3$$

$$f_{S4} = \frac{\partial U}{\partial u_4} = 0$$

$$k_{ij} = \int_0^L EA(x)\psi_i'(x)\psi_j'(x)dx \quad i,j = 1,3$$

$$\begin{Bmatrix} f_{S1} \\ f_{S2} \\ f_{S3} \\ f_{S4} \end{Bmatrix} = \begin{bmatrix} k_{11} & 0 & k_{13} & 0 \\ 0 & 0 & 0 & 0 \\ k_{31} & 0 & k_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\mathbf{f}_S^e = \mathbf{k}^e \mathbf{u}^e$$

$\mathbf{f}_S^e$ = nodal forces, $\mathbf{u}^e$ =nodal displacements and

$\mathbf{k}^e$ = stiffness matrix

$$\mathbf{k}^e = \begin{bmatrix} k_{11} & 0 & k_{13} & 0 \\ 0 & 0 & 0 & 0 \\ k_{31} & 0 & k_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $A(x) = A =$ constant

$$\mathbf{k}^e = \frac{EA^e}{L^e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (11.2.$$

ELEMENT KINETIC ENERGY AND MASS MATRIX

(a) Distributed mass. Consistent mass matrix

The mass element $m(x)dx$ undergoes two displacements $u(x,t)$ $v(x,t)$.

$$T=\frac{1}{2}\int_0^L m(x)\left\{\left[\dot{u}(x,t)\right]^2+\left[\dot{v}(x,t)\right]^2\right\}dx \qquad (11.2.20)$$

$$T(\dot{u}_1,\ldots,\dot{u}_4)=\frac{1}{2}\int_0^L m(x)\left\{\left[\dot{u}_1\psi_1(x)+\dot{u}_3\psi_3(x)\right]^2+\left[\dot{u}_2\psi_1(x)+\dot{u}_4\psi_3(x)\right]^2\right\}dx$$

$q_i=u_i,\ i=1,2,3,4$.

$$f_{I1}=\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}_1}\right)-\frac{\partial K}{\partial u_1}=m_{11}\ddot{u}_1+m_{13}\ddot{u}_3$$

$$f_{I2}=\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}_2}\right)-\frac{\partial K}{\partial u_2}=m_{22}\ddot{u}_2+m_{24}\ddot{u}_4$$

$$f_{I3}=\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}_3}\right)-\frac{\partial K}{\partial u_3}=m_{31}\ddot{u}_1+m_{33}\ddot{u}_3$$

$$f_{I4}=\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}_4}\right)-\frac{\partial K}{\partial u_4}=m_{42}\ddot{u}_2+m_{44}\ddot{u}_4$$

$$m_{ij}=m_{i+1,j+1}=\int_0^L m(x)\psi_i(x)\psi_j(x)dx\,,\ i,j=1,3$$

$$\begin{Bmatrix} f_{I1}\\ f_{I2}\\ f_{I3}\\ f_{I4}\end{Bmatrix}=\begin{bmatrix} m_{11} & 0 & m_{13} & 0\\ 0 & m_{22} & 0 & m_{24}\\ m_{31} & 0 & m_{33} & 0\\ 0 & m_{42} & 0 & m_{44}\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3\\ \ddot{u}_4\end{Bmatrix}$$

$$\mathbf{f}_I^e=\mathbf{m}^e\ddot{\mathbf{u}}^e \qquad (11.2.25)$$

$\mathbf{f}_I^e$ =nodal inertia forces,

$\ddot{\mathbf{u}}^e$ =nodal accelerations

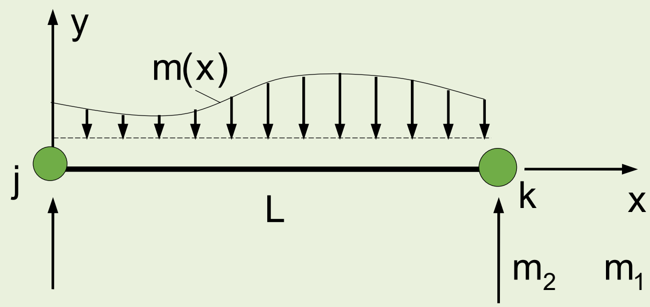
$\mathbf{m}^e$ =mass matrix (consistent)of element e

$$\mathbf{m}^e=\begin{bmatrix} m_{11} & 0 & m_{13} & 0\\ 0 & m_{22} & 0 & m_{24}\\ m_{31} & 0 & m_{33} & 0\\ 0 & m_{42} & 0 & m_{44}\end{bmatrix}$$

If $m(x)=\bar{m}$=constant

$$\mathbf{m}^e=\frac{m^e}{6}\begin{bmatrix} 2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\end{bmatrix},\quad m^e=\bar{m}^eL^e$$

(a) Lumped mass assumption



$$m_1 = \int_0^L m(x)(1-\xi)dx \qquad m_2 = \int_0^L m(x)\xi dx$$

$$T = \frac{1}{2}m_1\left(\dot{u}_1^2 + \dot{u}_2^2\right) + \frac{1}{2}m_2\left(\dot{u}_3^2 + \dot{u}_4^2\right)$$

Setting $q_i = u_i$, $i = 1,2,3,4$ we obtain

$$f_{I1} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_1}\right) - \frac{\partial T}{\partial u_1} = m_{11}\ddot{u}_1$$

$$f_{I2} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_2}\right) - \frac{\partial T}{\partial u_2} = m_{22}\ddot{u}_2$$

$$f_{I3} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_3}\right) - \frac{\partial T}{\partial u_3} = m_{33}\ddot{u}_3$$

$$f_{I4} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_4}\right) - \frac{\partial T}{\partial u_4} = m_{44}\ddot{u}_4$$

$$m_{11} = m_{22} = m_1, \qquad m_{33} = m_{44} = m_2$$

$$\begin{Bmatrix} f_{I1} \\ f_{I2} \\ f_{I3} \\ f_{I4} \end{Bmatrix} = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & 0 \\ 0 & 0 & 0 & m_{44} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{Bmatrix}$$

$$\mathbf{f}_I^e = \mathbf{m}^e \ddot{\mathbf{u}}^e$$

$\mathbf{f}_I^e$ =nodal inertia forces,

$\ddot{\mathbf{u}}^e$ =nodal accelerations

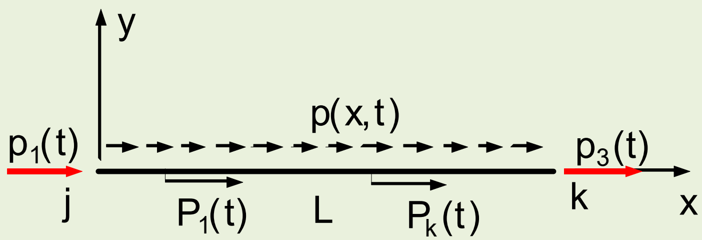
$\mathbf{m}^e$ =mass matrix of element e

$$\mathbf{m}^e = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & 0 \\ 0 & 0 & 0 & m_{44} \end{bmatrix}$$

If $m(x) = \bar{m}$ =constant

$$\mathbf{m}^e = \frac{1}{2}m^e \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad m^e = \bar{m}^e L^e$$

ELEMENT EXERNAL NODAL FORCES



$$\delta W_{nc}^{p} = \int_0^L p(x,t)\delta u(x,t)dx + \sum_{k=1}^{K} P_k \delta u(x_k,t)$$

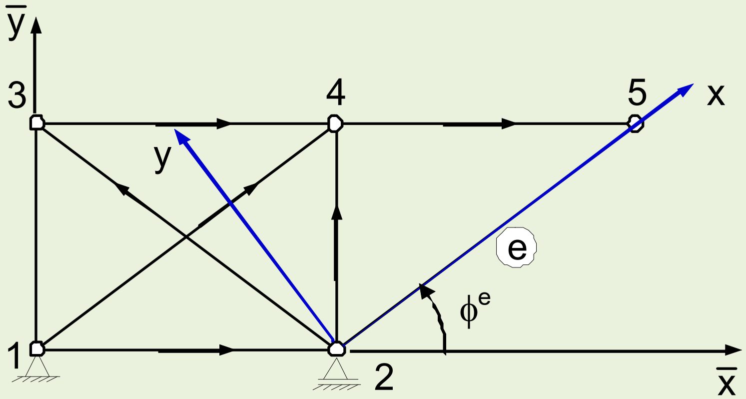
$$= \int_0^L p(x,t)\left[\delta u_1 \psi_1(x) + \delta u_3 \psi_3(x)\right]dx + \sum_{k=1}^{K} P_k(t)\left[\delta u_1 \psi_1(x_k) + \delta u_3 \psi_3(x_k)\right]$$

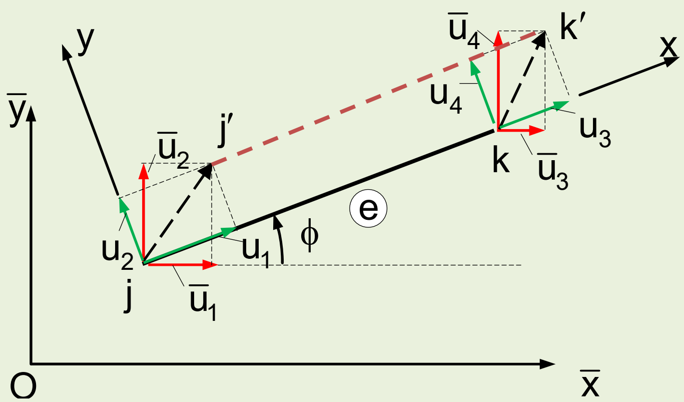
$$= p_1(t)\delta u_1 + p_3(t)\delta u_3$$

$$p_i(t) = \int_0^L p(x,t)\psi_i(x)dx + \sum_{k=1}^{K} P_k(t)\psi_i(x_k), \quad i = 1,3$$

$$\mathbf{p}^e(t) = \begin{Bmatrix} p_1(t) \\ 0 \\ p_3(t) \\ 0 \end{Bmatrix}$$

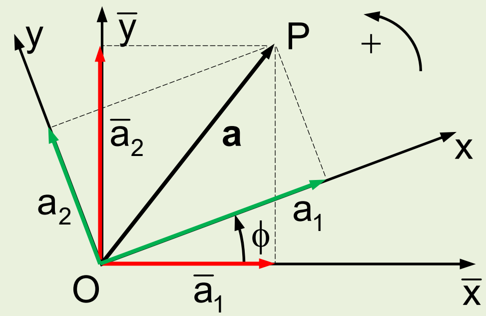
TRANFORMATION OF NODAL VECTORS AND ELEMENT MATRICES





$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{u}_4 \end{Bmatrix}$$



$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{Bmatrix}$$

$$\mathbf{a} = \boldsymbol{\Lambda}\bar{\mathbf{a}} \qquad \boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{T} \qquad \bar{\mathbf{a}} = \boldsymbol{\Lambda}^{T}\mathbf{a}$$

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \qquad \bar{\mathbf{a}} = \begin{Bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{Bmatrix}, \qquad \boldsymbol{\Lambda} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{Bmatrix}$$

$$\mathbf{u}^e = \mathbf{R}^e\bar{\mathbf{u}}^e \qquad \bar{\mathbf{u}}^e = \left(\mathbf{R}^e\right)^T \mathbf{u}^e$$

$$\bar{\mathbf{p}}^e(t) = \left(\mathbf{R}^e\right)^T \mathbf{p}^e(t)$$

$$\bar{\mathbf{f}}_S^e = \left(\mathbf{R}^e\right)^T \mathbf{f}_S^e$$

$$\bar{\mathbf{f}}_S^e = \left(\mathbf{R}^e\right)^T \mathbf{k}^e \mathbf{u}^e$$

$$\bar{\mathbf{f}}_S^e = \left(\mathbf{R}^e\right)^T \mathbf{k}^e \mathbf{R}^e \bar{\mathbf{u}}^e$$

$$\bar{\mathbf{f}}_S^e = \bar{\mathbf{k}}^e \bar{\mathbf{u}}^e \qquad \bar{\mathbf{k}}^e = (\mathbf{R}^e)^T \mathbf{k}^e \mathbf{R}^e$$

Similarly

$$\bar{\mathbf{f}}_I^e = \bar{\mathbf{m}}^e \ddot{\bar{\mathbf{u}}}^e \qquad \bar{\mathbf{m}}^e = (\mathbf{R}^e)^T \bar{\mathbf{m}}^e \mathbf{R}^e$$

The truss stiffness matrix, mass matrix and load vector

$$\bar{\mathbf{K}} = \sum_{e=1}^{N_e} \hat{\mathbf{K}}^e, \qquad \bar{\mathbf{M}} = \sum_{e=1}^{N_e} \hat{\mathbf{M}}^e$$

$$\bar{\mathbf{p}}(t) = \bar{\mathbf{P}}(t) + \sum_{e=1}^{N_e} \hat{\mathbf{p}}^e(t)$$

Equation of Motion

$$\bar{\mathbf{M}}\ddot{\bar{\mathbf{u}}} + \bar{\mathbf{K}}\bar{\mathbf{u}} = \bar{\mathbf{p}}(t)$$

The assembly matrix

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \end{Bmatrix} = \begin{bmatrix} 0 & \cdots & \underset{2j-1}{1} & \underset{2j}{0} & 0 & \cdots & 0 & \underset{2k-1}{0} & \underset{2k}{0} & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_N \end{Bmatrix}$$

$$\bar{\mathbf{u}} = \left\{ \bar{u}_1 \quad \bar{u}_2 \quad \cdots \quad \bar{u}_{2j-1} \quad \bar{u}_{2j} \quad \cdots \quad \bar{u}_{2k-1} \quad \bar{u}_{2k} \quad \cdots \quad \bar{u}_N \right\}^T$$

$$\bar{\mathbf{u}}^e = \mathbf{a}^e \bar{\mathbf{u}}, \qquad \bar{\mathbf{u}} = \left(\mathbf{a}^e\right)^T \bar{\mathbf{u}}^e$$

$$\hat{\mathbf{f}}_S^e = \left(\mathbf{a}^e\right)^T \bar{\mathbf{f}}_S^e$$

$$= \left(\mathbf{a}^e\right)^T \bar{\mathbf{k}}^e \bar{\mathbf{u}}^e$$

$$= \left(\mathbf{a}^e\right)^T \bar{\mathbf{k}}^e \mathbf{a}^e \bar{\mathbf{u}}$$

$$= \hat{\mathbf{K}}^e \bar{\mathbf{u}} \qquad \hat{\mathbf{K}}^e = \left(\mathbf{a}^e\right)^T \bar{\mathbf{k}}^e \mathbf{a}^e$$

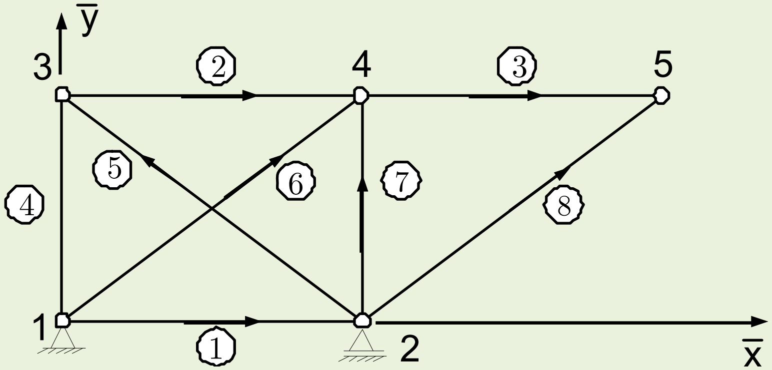
Similarly

$$\hat{\mathbf{f}}_I^e = \hat{\mathbf{M}}^e \ddot{\bar{\mathbf{u}}} \qquad \hat{\mathbf{M}}^e = \left(\mathbf{a}^e\right)^T \bar{\mathbf{m}}^e \mathbf{a}^e$$

The nodal forces

$$\hat{\mathbf{p}}^e(t) = \left(\mathbf{a}^e\right)^T \bar{\mathbf{p}}^e(t)$$

The FEM steps to formulate the equation of motion of a plane truss



Plane truss with $n=5$, $N=2n=10$, $N_e=8$.

(i) Idealization of the structure elements. A set of nodes interconnected with truss elements

- Selection of the global axes $\bar{x}\bar{y}$.
- Numbering of the nodes $(1,2,\dots,n)$ and determination of their Cartesian coordinates
- Determination of the vector $\bar{\mathbf{u}}$ $(N\times 1)$, $N=2n$
- Formulation of the vector $\bar{\mathbf{P}}(t)$, $(N\times 1)$ of the external forces acting directly on the nodes.
- Numbering of the elements $(1,2,\dots,N_e)$ and selection of their orientation (positive local axes x.

(ii) For each element $(e=1,2,\dots,N_e)$.

- Compute the matrices
  $\mathbf{k}^e$, $\mathbf{m}^e$, $\mathbf{p}^e(t)$ and $\mathbf{R}^e$
- Compute the matrices

$$\bar{\mathbf{k}}^e=\left(\mathbf{R}^e\right)^T\mathbf{k}^e\mathbf{R}^e, \qquad \bar{\mathbf{m}}^e=\left(\mathbf{R}^e\right)^T\mathbf{m}^e\mathbf{R}^e,$$

$$\bar{\mathbf{p}}^e(t)=\left(\mathbf{R}^e\right)^T\mathbf{p}^e(t)$$

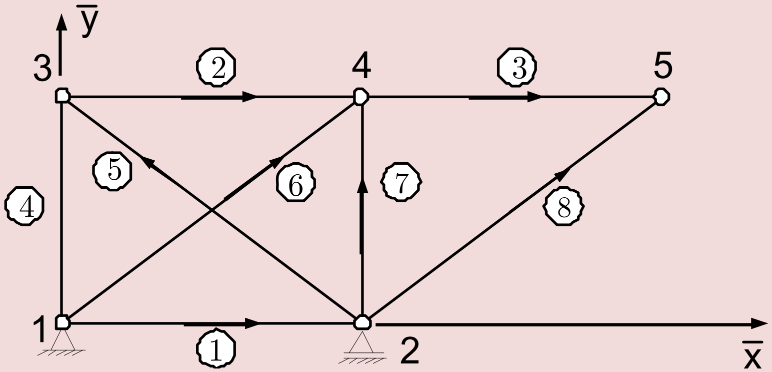
- Formulate the assembly matrices
  $\mathbf{a}^e$
- Formulate the enlarged matrices
  $\hat{\mathbf{K}}^e$, $\hat{\mathbf{M}}^e$, and $\hat{\mathbf{p}}^e(t)$

(iii) Compute the structure matrices.

$$\bar{\mathbf{K}}=\sum_{e=1}^{N_e}\hat{\mathbf{K}}^e, \qquad \bar{\mathbf{M}}=\sum_{e=1}^{N_e}\hat{\mathbf{M}}^e,$$

$$\bar{\mathbf{p}}(t)=\bar{\mathbf{P}}(t)+\sum_{e=1}^{N_e}\hat{\mathbf{p}}^e(t)$$

THE SUPPORTED STRUCTURE. MODIFICATION OF THE EQUATION OF MOTION



Plane truss with $n = 5$, $N = 2n = 10$, $N_e = 8$.

$$\tilde{\mathbf{u}}^T = \{\tilde{u}_1 \quad \tilde{u}_2 \quad \tilde{u}_3 \quad \tilde{u}_4 \quad \tilde{u}_5 \quad \tilde{u}_6 \quad \tilde{u}_7 \quad \tilde{u}_8 \quad \tilde{u}_9 \quad \tilde{u}_{10}\}$$
$$= \{\bar{u}_3 \quad \bar{u}_5 \quad \bar{u}_6 \quad \bar{u}_7 \quad \bar{u}_8 \quad \bar{u}_9 \quad \bar{u}_{10} \quad \bar{u}_1 \quad \bar{u}_2 \quad \bar{u}_4\}$$

Formulate the transformation matrix V (Boolean matrix)

$$\bar{\mathbf{u}} = \mathbf{V}\tilde{\mathbf{u}}, \qquad \mathbf{V}^{-1} = \mathbf{V}^T \quad \Rightarrow, \quad \tilde{\mathbf{u}} = \mathbf{V}^T\bar{\mathbf{u}}$$

$$\tilde{\mathbf{M}} = \mathbf{V}^T\bar{\mathbf{M}}\mathbf{V}$$

$$\tilde{\mathbf{K}} = \mathbf{V}^T\bar{\mathbf{K}}\mathbf{V}$$

$$\tilde{\mathbf{p}}(\mathbf{t}) = \mathbf{V}^T\bar{\mathbf{p}}(\mathbf{t})$$

s: Known degrees of freedom. They are specified by the support conditions.

$f = N - s$: Unknown degrees of freedom

Reorder the elements of $\bar{\mathbf{u}}$ so that

$$\tilde{u} = \begin{Bmatrix} \tilde{u}_f \\ \tilde{u}_s \end{Bmatrix} \frac{(f \times 1)}{(s \times 1)}$$

$$\begin{bmatrix} \tilde{\mathbf{M}}_{ff} & \tilde{\mathbf{M}}_{fs} \\ \tilde{\mathbf{M}}_{sf} & \tilde{\mathbf{M}}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{\mathbf{u}}}_f \\ \ddot{\tilde{\mathbf{u}}}_s \end{Bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ff} & \tilde{\mathbf{K}}_{fs} \\ \tilde{\mathbf{K}}_{sf} & \tilde{\mathbf{K}}_{ss} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}}_f \\ \tilde{\mathbf{u}}_s \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{p}}_f(t) \\ \tilde{\mathbf{p}}_s(t) \end{Bmatrix}$$

After performing the multiplications

$$\tilde{\mathbf{M}}_{ff}\ddot{\tilde{\mathbf{u}}}_f + \tilde{\mathbf{K}}_{ff}\tilde{\mathbf{u}}_f = \tilde{\mathbf{p}}_f^*(t)$$

$$\tilde{\mathbf{M}}_{sf}\ddot{\tilde{\mathbf{u}}}_f + \tilde{\mathbf{M}}_{ss}\ddot{\tilde{\mathbf{u}}}_s + \tilde{\mathbf{K}}_{sf}\tilde{\mathbf{u}}_f + \tilde{\mathbf{K}}_{ss}\tilde{\mathbf{u}}_s = \tilde{\mathbf{p}}_s(t)$$

where it was set

$$\tilde{\mathbf{p}}_f^*(t) = \tilde{\mathbf{p}}_f(t) - \tilde{\mathbf{M}}_{fs}\ddot{\tilde{\mathbf{u}}}_s - \tilde{\mathbf{C}}_{fs}\dot{\tilde{\mathbf{u}}}_s - \tilde{\mathbf{K}}_{fs}\tilde{\mathbf{u}}_s$$

Thank you for your Attention