NATIONAL TECHNICAL UNIVERSITY OF ATHENS
SCHOOL OF CIVIL ENGINEERING
Institute of Structural Analysis & Aseismic Research
Zografou Campus, GR.-157 73 Athens, GREECE

ΔΟΜΟΣΤΑΤΙΚΟΣ ΣΧΕΔΙΑΣΜΟΣ
ΚΑΙ ΑΝΑΛΥΣΗ ΤΩΝ ΚΑΤΑΣΚΕΥΩΝ
Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
Εθνικό Μετσόβιο Πολυτεχνείο

and

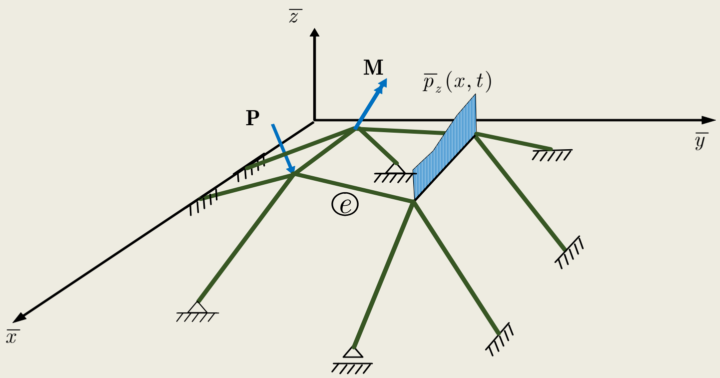
ADERS
MSc in Analysis and Design of Earthquake Resistant Structures

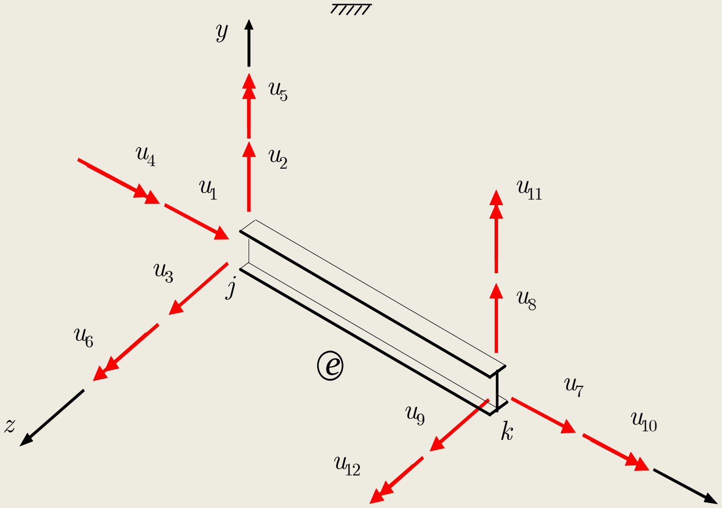
Advanced Dynamics of Structures

by J.T. Katsikadelis

Lecture # 10

THE FEM FOR SPACE FRAME





$$\mathbf{k}^e = \begin{bmatrix} \mathbf{k}^e_{jj} & \mathbf{k}^e_{jk} \\ \mathbf{k}^e_{kj} & \mathbf{k}^e_{kk} \end{bmatrix}, \qquad \mathbf{k}^e_{jk} = \left(\mathbf{k}^e_{kj}\right)^T$$

$$\mathbf{k}^e_{jj} = \frac{E}{L^3}\begin{bmatrix} L^2A & 0 & 0 & 0 & 0 & 0 \\ 0 & 12I_z & 0 & 0 & 0 & 6LI_z \\ 0 & 0 & 12I_y & 0 & -6LI_y & 0 \\ 0 & 0 & 0 & \frac{G}{E}L^2I_t & 0 & 0 \\ 0 & 0 & -6LI_y & 0 & 4L^2I_y & 0 \\ 0 & 6LI_z & 0 & 0 & 0 & 4L^2I_z \end{bmatrix}$$

$$\mathbf{k}^e_{kk} = \frac{E}{L^3}\begin{bmatrix} L^2A & 0 & 0 & 0 & 0 & 0 \\ 0 & 12I_z & 0 & 0 & 0 & -6LI_z \\ 0 & 0 & 12I_y & 0 & 6LI_y & 0 \\ 0 & 0 & 0 & \frac{G}{E}L^2I_t & 0 & 0 \\ 0 & 0 & 6LI_y & 0 & 4L^2I_y & 0 \\ 0 & -6LI_z & 0 & 0 & 0 & 4L^2I_z \end{bmatrix}$$

$$\mathbf{k}^e_{kj} = \frac{E}{L^3}\begin{bmatrix} -L^2A & 0 & 0 & 0 & 0 & 0 \\ 0 & -12I_z & 0 & 0 & 0 & -6LI_z \\ 0 & 0 & -12I_y & 0 & 6LI_y & 0 \\ 0 & 0 & 0 & -\frac{G}{E}L^2I_t & 0 & 0 \\ 0 & 0 & -6LI_y & 0 & 2L^2I_y & 0 \\ 0 & 6LI_z & 0 & 0 & 0 & 2L^2I_z \end{bmatrix}$$

$$\mathbf{m}^e = \begin{bmatrix} \mathbf{m}_{jj}^e & \mathbf{m}_{jk}^e \\ \mathbf{m}_{kj}^e & \mathbf{m}_{kk}^e \end{bmatrix}, \qquad \mathbf{m}_{jk}^e = \left(\mathbf{m}_{kj}^e\right)^T$$

Consistent mass matrix

$$\mathbf{m}_{jj}^e = \frac{m^e}{420}\begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 0 & 0 & 0 & 22L \\ 0 & 0 & 156 & 0 & -22L & 0 \\ 0 & 0 & 0 & 140r_g^2 & 0 & 0 \\ 0 & 0 & -22L & 0 & 4L^2 & 0 \\ 0 & 22L & 0 & 0 & 0 & 4L^2 \end{bmatrix}$$

$$\mathbf{m}_{kk}^e = \frac{m^e}{420}\begin{bmatrix} 140 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 0 & 0 & 0 & -22L \\ 0 & 0 & 156 & 0 & 22L & 0 \\ 0 & 0 & 0 & 140r_g^2 & 0 & 0 \\ 0 & 0 & 22L & 0 & 4L^2 & 0 \\ 0 & -22L & 0 & 0 & 0 & 4L^2 \end{bmatrix}$$

$$\mathbf{m}_{kj}^e = \frac{m^e}{420}\begin{bmatrix} 70 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 0 & 13L \\ 0 & 0 & 54 & 0 & -13L & 0 \\ 0 & 0 & 0 & 70r_g^2 & 0 & 0 \\ 0 & 0 & 13L & 0 & -3L^2 & 0 \\ 0 & -13L & 0 & 0 & 0 & -3L^2 \end{bmatrix}$$

$m^e = \rho AL$ is the total mass of the e element and $r_g = \sqrt{I_0 / A}$ is the radius of gyration of the cross-section.

Lumped mass matrix

$$\mathbf{m}^e_{jj} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{m}^e_{kk} = \begin{bmatrix} m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{m}^e_{jk} = \mathbf{m}^e_{kj} = \mathbf{0}$$

Nodal load vector

(a) Consistent nodal load vector

$$\mathbf{p}^e(t) = \begin{Bmatrix} \mathbf{p}^e_j \\ \mathbf{p}^e_k \end{Bmatrix}, \qquad \mathbf{p}^e_j = \{p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6\}^T, \qquad \mathbf{p}^e_k = \{p_7 \quad p_8 \quad p_9 \quad p_{10} \quad p_{11} \quad p_{12}\}^T$$

(b) Statically equivalent nodal load vector

$$\mathbf{p}^e(t) = \begin{Bmatrix} \mathbf{p}^e_j \\ \mathbf{p}^e_k \end{Bmatrix} \qquad \mathbf{p}^e_j(t) = \{p_1 \quad p_2 \quad p_3 \quad p_4 \quad 0 \quad 0\}^T, \qquad \mathbf{p}^e_k = \{p_7 \quad p_8 \quad p_9 \quad p_{10} \quad 0 \quad 0\}^T$$

Transformation of nodal vectors from local to global axes

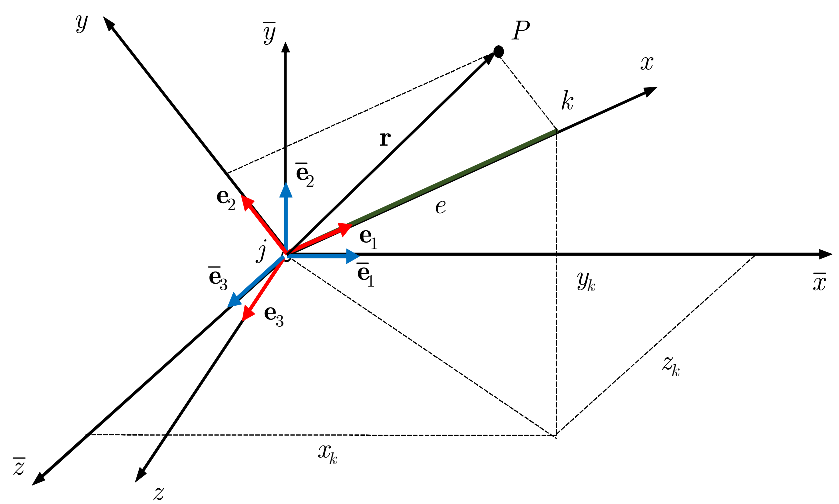


Figure. Local and global axes of spaceframe element

$$\mathbf{e}_1 = \lambda_{11}\bar{\mathbf{e}}_1 + \lambda_{12}\bar{\mathbf{e}}_2 + \lambda_{13}\bar{\mathbf{e}}_3$$

$$\mathbf{e}_2 = \lambda_{21}\bar{\mathbf{e}}_1 + \lambda_{22}\bar{\mathbf{e}}_2 + \lambda_{23}\bar{\mathbf{e}}_3$$

$$\mathbf{e}_3 = \lambda_{31}\bar{\mathbf{e}}_1 + \lambda_{32}\bar{\mathbf{e}}_2 + \lambda_{33}\bar{\mathbf{e}}_3$$

$$\bar{\mathbf{r}} = (\bar{x}_p - \bar{x}_j)\bar{\mathbf{e}}_1 + (\bar{y}_p - \bar{y}_j)\bar{\mathbf{e}}_2 + (\bar{z}_p - \bar{z}_j)\bar{\mathbf{e}}_3$$

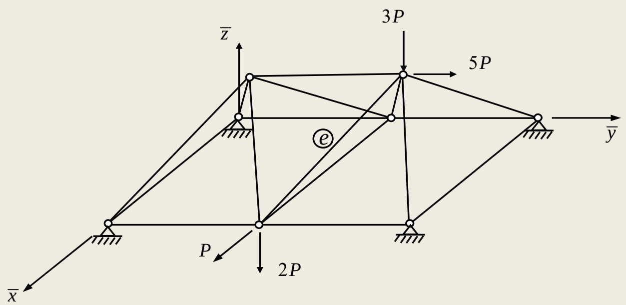
$$\lambda_{11} = \frac{\bar{x}_k - \bar{x}_j}{L}, \qquad \lambda_{12} = \frac{\bar{y}_k - \bar{y}_j}{L}, \qquad \lambda_{13} = \frac{\bar{z}_k - \bar{z}_j}{L}, \qquad \mathbf{e}_3 = \frac{\mathbf{e}_1 \times \bar{\mathbf{r}}}{|\mathbf{e}_1 \times \bar{\mathbf{r}}|}, \qquad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$$

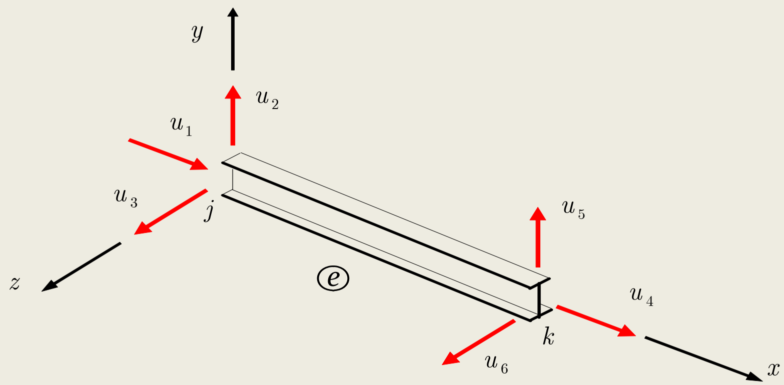
$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \qquad \boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{T}$$

Transformation matrix

$$\mathbf{R}^{e} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda} \end{bmatrix}$$

THE FEM FOR SPACE TRUSS





Displacements

$$\mathbf{u} = \left\{ u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \right\}^T$$

Elastic energy

$$U(u_1, u_2, \dots, u_6) = \frac{1}{2}\int_0^L EA(x)\left[u'(x,t)\right]^2 dx$$

Kinetic energy for lumped mass assumption

$$T(\dot{u}_1, \dot{u}_2, \dots, \dot{u}_6) = \frac{1}{2}m_1\left(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2\right) + \frac{1}{2}m_2\left(\dot{u}_4^2 + \dot{u}_5^2 + \dot{u}_6^2\right)$$

For $A(x) = A = \text{consatnt}$

$$\mathbf{k}^e = \frac{EA^e}{L^e}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{m}^e = \begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{66} \end{bmatrix}$$

where $m_{11} = m_{22} = m_{33} = m_1$, $m_{44} = m_{55} = m_{66} = m_2$ και

$$\mathbf{p}^e(t) = \left\{ p_1(t) \quad 0 \quad 0 \quad p_4(t) \quad 0 \quad 0 \right\}^T$$

Transformation of nodal vectors from local to global axes

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix}, \quad \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{Bmatrix} \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \end{Bmatrix}$$

which are combined to

$$\bar{\mathbf{u}}^e = \left(\mathbf{R}^e\right)^T \mathbf{u}^e$$

where

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{bmatrix}$$

is the transformation matrix of the e element of the space truss with dimensions 6×6. The remaining procedure for formulating the equation of motion is the same as for the plane truss.

Free vibrations of multi-degree-of-freedom systems

FREE VIBRATIONS WITHOUT DAMPING

$\mathbf{M\ddot{u}+C\dot{u}+Ku=0}$ $\quad\quad$ $\mathbf{C=0}$ $\quad\quad$ $\mathbf{M\ddot{u}+Ku=0}$

Look for a solution in the form

$$u_1=\beta_1 T(t),\ u_2=\beta_2 T(t),\ldots,u_N=\beta_N T(t)$$

$\beta_1,\beta_2,\ldots,\beta_N$:=are constants $T(t)$:= common function of time

$$\mathbf{u}=\boldsymbol{\beta}T(t) \qquad \boldsymbol{\beta}=\{\beta_1,\beta_2,\ldots,\beta_N\}^T$$

$$\mathbf{M}\boldsymbol{\beta}\ddot{T}(t)+\mathbf{K}\boldsymbol{\beta}T(t)=\mathbf{0}$$

Premultiplying with $\boldsymbol{\beta}^T$ gives

$$\boldsymbol{\beta}^T\mathbf{M}\boldsymbol{\beta}\ddot{T}(t)+\boldsymbol{\beta}^T\mathbf{K}\boldsymbol{\beta}T(t)=\mathbf{0} \qquad (12.2.4)$$

excluding $T(t)=0$

$$-\frac{\ddot{T}(t)}{T(t)}=\frac{\boldsymbol{\beta}^T\mathbf{K}\boldsymbol{\beta}}{\boldsymbol{\beta}^T\mathbf{M}\boldsymbol{\beta}}=\lambda=\text{constant}$$

$$\ddot{T}(t)+\lambda T(t)=0$$

(i) $\lambda=0.$ $\quad$ $\ddot{T}(t)=0$

$T(t)=C_1 t+C_2$ no vibration

(ii) $\lambda=-\omega^2<0.$ $\quad$ $\ddot{T}(t)-\omega^2 T(t)=0$

$T(t)=C_1 e^{\omega t}+C_2 e^{-\omega t}$ no vibration

(iii) $\lambda=\omega^2>0.$ $\quad$ $\ddot{T}(t)+\omega^2 T(t)=0$

$$T(t)=c\cos\omega t+d\sin\omega t = a\cos(\omega t-\theta)$$ vibration

$$\ddot{T}(t)=-\omega^2 T(t) \qquad \left(\mathbf{K}-\omega^2\mathbf{M}\right)\boldsymbol{\beta}T(t)=\mathbf{0}$$

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\boldsymbol{\beta}=\mathbf{0}$$

Linear eigenvalue

$$\det(\mathbf{K}-\omega^2\mathbf{M})=\mathbf{0}$$

SOLUTION OF THE EIGENVALUE PROBLEM

$$\det\left(\mathbf{K}-\omega^2\mathbf{M}\right)=\begin{vmatrix} k_{11}-\omega^2 m_{11} & k_{12}-\omega^2 m_{12} & \cdots & k_{1N}-\omega^2 m_{1N} \\ k_{21}-\omega^2 m_{21} & k_{22}-\omega^2 m_{22} & \cdots & k_{2N}-\omega^2 m_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ k_{N1}-\omega^2 m_{N1} & k_{N2}-\omega^2 m_{N2} & \cdots & k_{NN}-\omega^2 m_{NN} \end{vmatrix}=0$$

Expand the determinant

$$\Pi(\lambda)=\alpha_0\lambda^N+\alpha_1\lambda^{N-1}+\cdots+\alpha_{N-1}\lambda\ +\alpha_N=0, \qquad \lambda=\omega^2 \qquad (12.5.10)$$

Due to properties of $\mathbf{M}$, $\mathbf{K}$ the roots are real and positive

$\omega_1<\omega_2\ldots<\omega_N$, $\qquad\qquad$ ω_1:= fundamental eigenfrequency

$$\mathbf{A}(\omega_i)\mathbf{B}=\left[\begin{array}{c|cccc} k_{11}-\omega_i^2 m_{11} & k_{12}-\omega_i^2 m_{12} & \cdots & k_{1N}-\omega_i^2 m_{1N} \\ \hline k_{21}-\omega_i^2 m_{21} & k_{22}-\omega_i^2 m_{22} & \cdots & k_{2N}-\omega_i^2 m_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ k_{N1}-\omega_i^2 m_{N1} & k_{N2}-\omega_i^2 m_{N2} & \cdots & k_{NN}-\omega_i^2 m_{NN} \end{array}\right]\left\{\begin{array}{c} 1 \\ \hline \beta_2^{(i)} \\ \cdots \\ \beta_N^{(i)} \end{array}\right\}=\left\{\begin{array}{c} 0 \\ \hline 0 \\ \cdots \\ 0 \end{array}\right\}$$

$$\mathbf{A}_{11}^{(i)}=\left[k_{11}-\omega_i^2 m_{11}\right]_{1\times 1} \qquad \mathbf{A}_{12}^{(i)}=\left[k_{12}-\omega_i^2 m_{12} \quad \cdots \quad k_{1N}-\omega_i^2 m_{1N}\right]_{1\times(N-1)}$$

$$\mathbf{A}_{21}^{(i)}=\begin{bmatrix} k_{21}-\omega^2 m_{21} \\ \cdots \\ k_{n1}-\omega_i^2 m_{N1} \end{bmatrix}_{(N-1)\times 1} \qquad \mathbf{A}_{22}^{(i)}=\begin{bmatrix} k_{22}-\omega_i^2 m_{22} & \cdots & k_{2N}-\omega_i^2 m_{2N} \\ \cdots & \cdots & \cdots \\ k_{N2}-\omega_i^2 m_{N2} & \cdots & k_{NN}-\omega_i^2 m_{NN} \end{bmatrix}_{(N-1)\times(N-1)}, \qquad \boldsymbol{\beta}_2^{(i)}=\begin{Bmatrix} \beta_2^{(i)} \\ \cdots \\ \beta_N^{(i)} \end{Bmatrix}_{(N-1)\times 1}$$

Partitioning

$$\begin{bmatrix} \mathbf{A}_{11}^{(i)} & \mathbf{A}_{12}^{(i)} \\ \mathbf{A}_{21}^{(i)} & \mathbf{A}_{22}^{(i)} \end{bmatrix} \begin{Bmatrix} 1 \\ \boldsymbol{\beta}_2^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \mathbf{0} \end{Bmatrix} \qquad \begin{matrix} \mathbf{A}_{11}^{(i)} + \mathbf{A}_{12}^{(i)} \boldsymbol{\beta}_2^{(i)} = 0 \\ \\ \mathbf{A}_{21}^{(i)} + \mathbf{A}_{22}^{(i)} \boldsymbol{\beta}_2^{(i)} = \mathbf{0} \end{matrix} \, , \qquad \boldsymbol{\beta}_2^{(i)} = -\left[\mathbf{A}_{22}^{(i)} \right]^{-1} \mathbf{A}_{21}^{(i)}$$

$$\mathbf{B} = [\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \cdots \quad \boldsymbol{\beta}_N] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{NN} \end{bmatrix} \quad \text{eigenvectors}$$

$$\begin{aligned} \mathbf{u}_i &= \boldsymbol{\beta}_i (c_i \cos \omega_i t + d_i \sin \omega_i t) \\ &= \boldsymbol{\beta}_i a_i \cos(\omega t - \theta_i) \end{aligned}$$

$$\mathbf{u} = \sum_{i=1}^{N} \boldsymbol{\beta}_i (c_i \cos \omega_i t + d_i \sin \omega_i t)$$

or

$$\mathbf{u} = \sum_{i=1}^{N} \boldsymbol{\beta}_i a_i \cos(\omega_i t - \theta_i) \qquad (12.2.26)$$

NORMALIZATION

With respect to the maximum element $\phi_i = \begin{bmatrix} \phi_{1i} \\ \phi_{2i} \\ \cdots \\ \phi_{Ni} \end{bmatrix} = \dfrac{1}{\max \beta_{ki}} \begin{bmatrix} 1 \\ \beta_{2i} \\ \cdots \\ \beta_{Ni} \end{bmatrix}$ mode shapes

$$\boldsymbol{\Phi} = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_N] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix}$$

Another method of normalization: With respect to the mass

$\phi_i^T \mathbf{M} \phi_i = 1$ ϕ_i represents the normalized eigenvector $\phi_i = \mu_i \boldsymbol{\beta}_i$

Substitution gives

$1 = \phi_i^T \mathbf{M} \phi_i = \mu_i^2 \beta_i^T \mathbf{M} \beta_i$ $\mu_i = \dfrac{1}{\sqrt{\beta_i^T \mathbf{M} \beta_i}}$ Hence $\phi_i = \dfrac{\boldsymbol{\beta}_i}{\sqrt{\beta_i^T \mathbf{M} \beta_i}}$

If $\mathbf{M} = \mathbf{I}$, $\sqrt{\beta_i^T \mathbf{I} \beta_i} = \sqrt{\beta_i^T \beta_i}$ is the magnitude of β_i, that is, ϕ_i is orthonormal

Mathematical study of the eigenvalue problem and methods of establishing the eigenvalues and eigenvectors are presented in Chapter 12 and Chapter 13

COMPUTATION OF EINGENFREQUENCIES AND MODE SHAPES

Linear eigenvalue $\det(\mathbf{K} - \lambda \mathbf{M}) = 0 \qquad \lambda = \omega^2$

Rayleigh quotient to approximate the first eigenfrequency

$$\rho(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} \qquad \lambda = \rho(\mathbf{x})$$

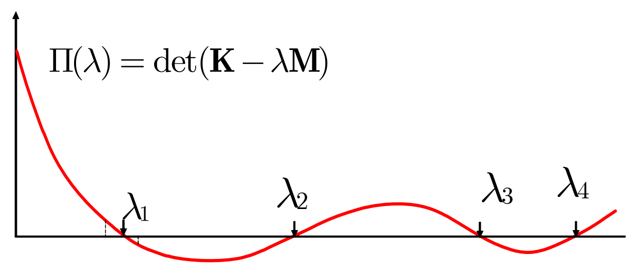
1. Transformation methods:

(i) Jacobi diagonalization, 1846, (ii) Givens triangularization (ii) Householder transformation, and (iv) QR and QL transformations

2. Vector iteration methods:

(i) Vector iteration, (ii) Vector iteration with shift (iii) Subspace iteration and (iv) Lanczos method.

3. Determinant search method; Find the roots of $\Pi(\lambda) = \det(\mathbf{K} - \lambda \mathbf{M})$



Use of IMSL, MATLAB, MAPLE, MATHEMATICA to evaluate the eigenvalues and mode shapes

INVERSE VECTOR ITERATION METHOD (Vianello 1898 - Stodola 1904)

1. Assume a $\mathbf{x}_k$. We determine the vector $\bar{\mathbf{x}}_{k+1}$ by solving the linear system

$$\mathbf{K}\bar{\mathbf{x}}_{k+1} = \mathbf{M}\mathbf{x}_k \tag{12.2.4}$$

2. We compute the approximate value of the eigenvalue $\lambda^{(k+1)}$ corresponding to $\bar{\mathbf{x}}_{k+1}$ using the Rayleigh quotient [Eq.(11.6.21)]

$$\rho(\bar{\mathbf{x}}_{k+1}) = \frac{\bar{\mathbf{x}}_{k+1}^T \mathbf{K}\bar{\mathbf{x}}_{k+1}}{\bar{\mathbf{x}}_{k+1}^T \mathbf{M}\bar{\mathbf{x}}_{k+1}}, \qquad \lambda^{(k+1)} = \rho(\bar{\mathbf{x}}_{k+1}) \tag{12.2.5}$$

3. We check for convergence by comparing two consecutive values of λ

$$\frac{\left|\lambda^{(k+1)} - \lambda^{(k)}\right|}{\lambda^{(k+1)}} < \varepsilon, \qquad \text{where } \varepsilon \text{ is a specified tolerance} \tag{12.2.6}$$

4 If the convergence criterion is not satisfied, we normalize $\bar{\mathbf{x}}_{k+1}$ with respect to mass

$$\mathbf{x}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1}}{\left(\bar{\mathbf{x}}_{k+1}^T \mathbf{M}\bar{\mathbf{x}}_{k+1}\right)^{1/2}} \tag{12.2.7}$$

and go back to step 1.

5. If the criterion is satisfied the procedure stops and we set

$$\phi_1 \simeq \mathbf{x}_{k+1}, \qquad \lambda_1 \simeq \lambda^{(k+1)} \tag{12.2.8}$$

ORTHOGONALITY OF MODE SHAPES

$$\mathbf{K}\beta_n = \omega_n^2 \mathbf{M}\beta_n$$

$$\mathbf{K}\beta_i = \omega_i^2 \mathbf{M}\beta_i$$

After normalization

$$\mathbf{K}\phi_n = \omega_n^2 \mathbf{M}\phi_n \qquad (1)$$

$$\mathbf{K}\phi_i = \omega_i^2 \mathbf{M}\phi_i \qquad (2)$$

Premultiply (1) by ϕ_i^T

$$\phi_i^T \mathbf{K}\phi_n = \omega_n^2 \phi_i^T \mathbf{M}\phi_n \qquad \text{transpose}$$

$$\phi_n^T \mathbf{K}^T \phi_i = \omega_n^2 \phi_n^T \mathbf{M}^T \phi_i \qquad \mathbf{K}^T = \mathbf{K}, \qquad \mathbf{M}^T = \mathbf{M} \qquad \phi_n^T \mathbf{K}\phi_i = \omega_n^2 \phi_n^T \mathbf{M}\phi_i \quad (3)$$

Premultiply (2) by ϕ_n^T

$$\phi_n^T \mathbf{K}\phi_i = \omega_i^2 \phi_n^T \mathbf{M}\phi_i \quad (4)$$

Substract (3) and (4)

$$\left(\omega_n^2 - \omega_i^2\right)\phi_n^T \mathbf{M}\phi_i = 0 \qquad (12.3.9)$$

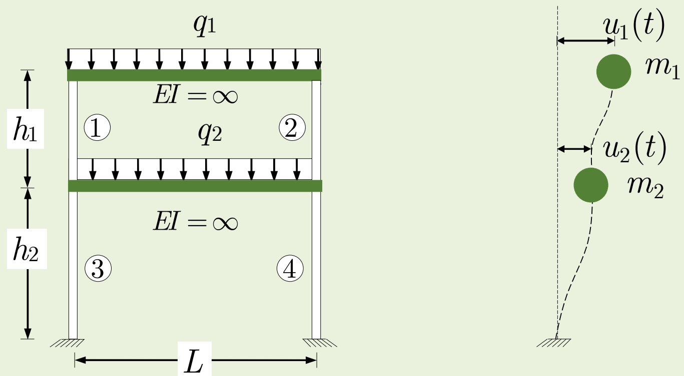
if $\omega_i \neq \omega_n$

$$\phi_n^T \mathbf{M}\phi_i = 0$$

$$\phi_n^T \mathbf{K}\phi_i = \omega_n^2 \phi_n^T \mathbf{M}\phi_i = 0$$

For multiple eigenfrequencies see Chapter 12

EXAMPLE



(a) (b)

Data: $h_2=4.0\,\mathrm{m}$, $q_1=40\,\mathrm{kN/m}$, $q_2=50\,\mathrm{kN/m}$, $L=6.0\,\mathrm{m}$. Cross section of columns: 1st story $25\times25\,\mathrm{cm}^2$, Ground level $30\times30\,\mathrm{cm}^2$, $\gamma_b=24\,\mathrm{kN/m}^3$ και $E_b=2.1\times10^7\,\mathrm{kN/m}^2$. q_1, q_2 include the self weight of beams. Lumped mass assumption.

(i) Mass matrix **M**.

$$m_{11}=\frac{6.0\times40+2\times(0.25\times0.25\times\frac{3.5}{2})\times24}{9.81}$$
$$=25\,\mathrm{kNm^{-1}sec^2}$$

$$m_{22}=\frac{6.0\times50+2\times(0.25\times0.25\times\frac{3.5}{2}+0.30\times0.30\times\frac{4.0}{2})\times24}{9.81}$$
$$=32\,\mathrm{kNm^{-1}sec^2}$$

Hence

$$\mathbf{M}=\begin{bmatrix}25 & 0\\ 0 & 32\end{bmatrix}$$

(ii) Stiffness matrix **K**

$$k_{11}=k_{11}^{(1)}+k_{11}^{(2)}=2\frac{12EI_1}{h_1^3}$$
$$=2\frac{12\times2.1\times10^7\times0.25^4}{3.5^3\times12}=3826.5\,\mathrm{kN/m}$$

$$k_{21} = k_{31}^{(1)} + k_{31}^{(2)} = -2\frac{12EI_1}{h_1^3}$$

$$= -2\frac{12 \times 2.1 \times 10^7 \times 0.25^4}{3.5^3 \times 12} = -3826.5\,\text{kN / m}$$

$$k_{22} = k_{33}^{(1)} + k_{33}^{(1)} + k_{11}^{(3)} + k_{11}^{(4)} = 2\frac{12EI_1}{h_1^3} + 2\frac{12EI_3}{h_2^3}$$

$$= 2\frac{12 \times 2.1 \times 10^7 \times 0.25^4}{3.5^3 \times 12} + 2\frac{12 \times 2.1 \times 10^7 \times 0.30^4}{4^3 \times 12} = 9142.1\,\text{kN / m}$$

Hence

$$\mathbf{K} = \begin{bmatrix} 3826.5 & -3826.5 \\ -3826.5 & 9142.1 \end{bmatrix}$$

(iii) Eigen frequencies

$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = \begin{vmatrix} 3826.5 - 25\omega^2 & -3826.5 \\ -3826.5 & 9142.1 - 32\omega^2 \end{vmatrix} = 0 \qquad (3)$$

Expanding the determinant

$$\omega^4 - 438.7506\omega^2 + 25425.1792 = 0 \qquad (4)$$

$$\omega_1^2 = 68.7089, \qquad \omega_2^2 = 370.0416$$

$$\omega_1 = 8.289, \qquad \omega_2 = 19.236 \qquad (5)$$

(iv) Mode shapes

$$(3826.5-25\omega_i^2)-3826.5\beta_{2i}=0$$

$$-3826.5+(9142.1-32\omega_i^2)\beta_{2i}=0$$

για $i=1$

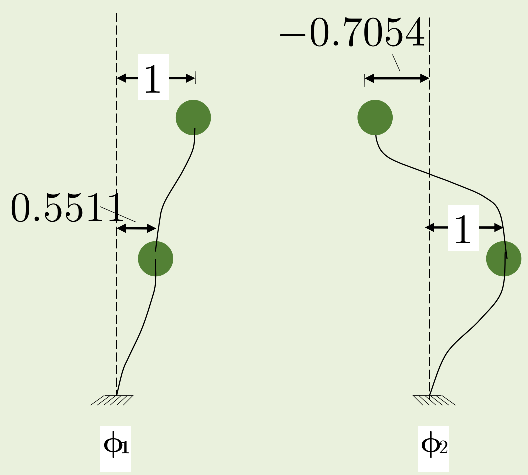
$$\beta_{21}=(3826.5-25\times 8.289^2)/3826.5=0.5511$$

για $i=2$

$$\beta_{22}=(3826.5-25\times 19.236^2)/3826.5=-1.41751$$

Eigenvectors

$$\mathbf{B}=\begin{bmatrix}1 & 1\\ 0.5511 & -1.41751\end{bmatrix}$$



(i) Normalization with respect to the largest element

$$\mathbf{\Phi}=\begin{bmatrix}1 & -0.7054\\ 0.5511 & 1\end{bmatrix} \qquad (7)$$

(ii) Normalization with respect to mass

$$\phi_1=\frac{\beta_1}{\sqrt{\beta_1^T M\beta_1}}$$

$$=\frac{1}{\sqrt{\{1 \;\; 0.5511\}\begin{bmatrix}25 & 0\\ 0 & 32\end{bmatrix}\begin{Bmatrix}1\\ 0.5511\end{Bmatrix}}}\begin{Bmatrix}1\\ 0.5511\end{Bmatrix}=\begin{Bmatrix}0.1697\\ 0.0935\end{Bmatrix}$$

$$\phi_2=\frac{\beta_2}{\sqrt{\beta_2^T M\beta}}$$

$$=\frac{1}{\sqrt{\{1 \;\; -1.41276\}\begin{bmatrix}25 & 0\\ 0 & 32\end{bmatrix}\begin{Bmatrix}1\\ -1.41276\end{Bmatrix}}}\begin{Bmatrix}1\\ -1.41276\end{Bmatrix}=\begin{Bmatrix}0.1061\\ -0.1499\end{Bmatrix}$$

Hence

$$\mathbf{\Phi}=\begin{bmatrix}0.1697 & 0.1061\\ 0.0935 & -0.1499\end{bmatrix} \qquad (8)$$

Properties of the eigenfrequencies and modes shapes of free vibrations

If $\mathbf{K}$, $\mathbf{M}$ real and symmetric, all EFs (eigenfrequencies) are real.

If $\mathbf{K}$, $\mathbf{M}$ real, symmetric and positive definite $U(\mathbf{u}) = \mathbf{u}^T\mathbf{K}\mathbf{u} > 0$, all EFs are real and positive.

If $\mathbf{K}$ singular, $\det(\mathbf{K}) = 0$, at least one EF is zero and the respective mode shape expresses rigid body motion. The Elastic energy is positive semi-definite, $U(\mathbf{u}) = \mathbf{u}^T\mathbf{K}\mathbf{u} \geq 0$ for $\mathbf{u} \neq 0$.

If $\mathbf{M}$ singular, $\det(\mathbf{M}) = 0$, at least one EF is infinite and the kinetic energy is positive semi-definite, $T(\dot{\mathbf{u}}) = \dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} \geq 0$ for $\dot{\mathbf{u}} \neq 0$.

The number of mode shapes corresponding to an EF is equal to its multiplicity.

The mode shapes are linear independent and orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$.

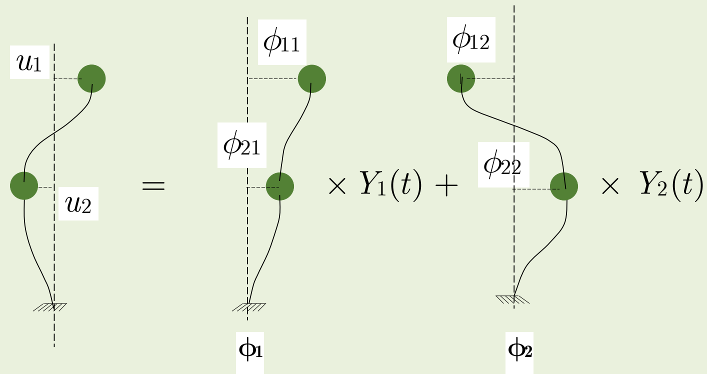
$$\phi_m^T\mathbf{M}\phi_n = 0, \; m \neq n$$

$$\phi_m^T\mathbf{K}\phi_n = 0, \; m \neq n$$

Any vector can be represented as a superposition of the mode shapes.

$$\mathbf{u} = \mathbf{\Phi}\mathbf{a}, \quad \mathbf{a} = \mathbf{\Phi}^T\mathbf{u}$$

THE METHOD OF MODE SUPERPOSITION



$$\mathbf{u}(t) = \phi_1 Y_1(t) + \phi_2 Y_2(t) + \cdots + \phi_N Y_N(t)$$
$$= \mathbf{\Phi Y}(t) \qquad (12.9.1)$$

$$\mathbf{M}(\phi_1 \ddot{Y}_1 + \phi_2 \ddot{Y}_2 + \cdots + \phi_N \ddot{Y}_N) + \mathbf{K}(\phi_1 Y_1 + \phi_2 Y_2 + \cdots + \phi_N Y_N) = \mathbf{0}$$

Premultiply with ϕ_n^T and use the orthogonality conditions

$\phi_n^T \mathbf{M} \phi_j = 0 \quad \phi_n^T \mathbf{K} \phi_j = 0$ for $n \neq j$

$$\phi_n^T \mathbf{M} \phi_n \ddot{Y}_n + \phi_n^T \mathbf{K} \phi_n Y_n = \mathbf{0}, \qquad n = 1, 2, \ldots, N$$

We set

$$M_n = \phi_n^T \mathbf{M} \phi_n \qquad (12.9.4)$$
$$K_n = \phi_n^T \mathbf{K} \phi_n \qquad (12.9.5)$$

then

$$M_n \ddot{Y}_n + K_n Y_n = \mathbf{0}, \qquad n = 1, 2, \ldots, N \quad (12.9.6)$$

$$K_n = \omega_n^2 M_n \quad (12.9.7)$$
$$\ddot{Y}_n + \omega_n^2 Y_n = 0, \qquad n = 1, 2, \ldots, N \quad (12.9.8)$$

N uncoupled one degree-of-freedom systems

$$Y_n = \frac{\dot{Y}_n(0)}{\omega_n} \sin \omega_n t + Y_n(0) \cos \omega_n t, \quad n = 1, 2, \ldots, N$$

Initial conditions $Y_n(0), \dot{Y}_n(0)$ are expressed in terms of $\mathbf{u}(0), \dot{\mathbf{u}}(0)$.

Premultiply (12.9.1) with $\phi_n^T \mathbf{M}$ gives

$$Y_n(t) = \frac{\phi_n^T \mathbf{M} \mathbf{u}(t)}{M_n} \qquad n = 1, 2, \ldots, N$$
$$\dot{Y}_n(t) = \frac{\phi_n^T \mathbf{M} \dot{\mathbf{u}}(t)}{M_n} \qquad n = 1, 2, \ldots, N$$

for $t = 0$ we have

$$Y_n(0) = \frac{\phi_n^T \mathbf{M} \mathbf{u}(0)}{M_n} \qquad n = 1, 2, \ldots, N$$
$$\dot{Y}_n(0) = \frac{\phi_n^T \mathbf{M} \dot{\mathbf{u}}(0)}{M_n} \qquad n = 1, 2, \ldots, N$$

$\mathbf{u}_n = \phi_n Y_n, \; n = 1, 2, \ldots, N,$

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_N \qquad (12.9.18)$$

FREE VIBRATIONS WITH DAMPING

$$\mathbf{M\ddot{u} + C\dot{u} + Ku = 0} \qquad (12.10.1)$$

$$\mathbf{u} = \beta e^{\lambda t} \qquad (12.10.2)$$

$$\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right)\beta = \mathbf{0} \qquad (12.10.3)$$

Quadratic eigenvalue problem

$$\mathbf{S}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} \qquad (12.10.4)$$

$$\det\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right) = 0 \qquad (12.10.5)$$

Polynomial of $2N$ degree with respect to λ. Characteristic equation

$$\Pi(\lambda) = a_0\lambda^{2N} + a_1\lambda^{2N-1} + \cdots + a_{2N} = 0 \qquad (12.10.6)$$

$2N$ roots $\lambda_1, \lambda_2, \ldots, \lambda_{2N}$. The coefficients are real

$$\mathbf{u} = \sum_{n=1}^{2N} a_n \beta_n e^{\lambda_n t} \qquad (12.10.7)$$

a_n are $2N$ arbitrary constants determined from the initial conditions

Thank you for your Attention