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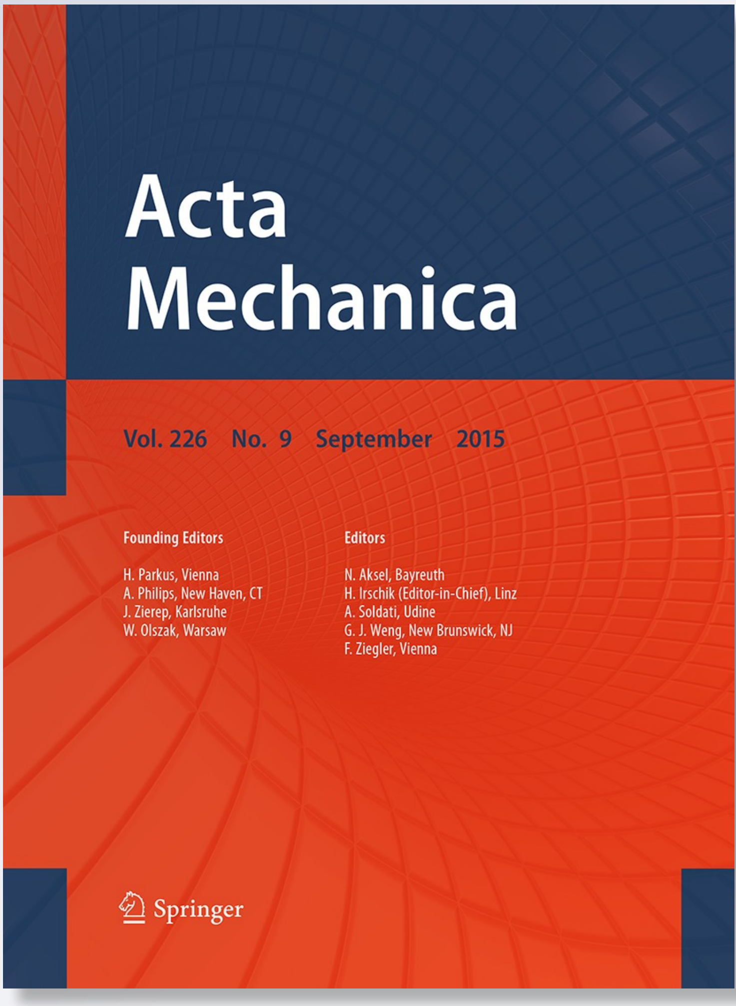

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Abstract Newton's law of motion is derived using Galileo's experimental data from the inclined plane. This is achieved by developing a simple system identification method using measured distances and corresponding times on the inclined plane and the concepts of the derivative and differential equation. The identification procedure yields the differential equation of motion of systems with constant as well as variable mass. As the employed mathematical tools were available before 1686, we are allowed to state that the equation of motion could have been established using only Galileo's experimental data, before Newton had published his law of motion. The differential equation of motion for systems with mass ejection (rocket equation) or mass accretion is also derived.

1 Introduction

Newton's law of motion (second law) was presented in his book "Philosophiae Naturalis Principia Mathematica" published on July 5, 1686 [1]. This law in the original Latin text reads [1, p. 12]:

> **Lex II.**
> **Mutationem motus propotionalem esse vi motrici impressae, ac fieri secundum lineam rectam qua vis illa imprimitur.**

which is translated in English as [2, p. 83]

> Law II.
> The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

By **motus** (motion) or **quantitas motus** (quantity of motion), Newton defined the product of the mass (**quantitas materiae**) times the velocity (**velocitas**) [1, Def. I and II. p. 2], i.e., what we refer to as *momentum*. Thus, in mathematical language, Newton's second law of motion can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = f \tag{1}$$

where m is the mass, v the velocity and $f = f(t)$ the external force.

Newton presented this law as an *axiom* (**Axiomata sive Leges Motus, Lex II** [1, p. 12]). He did not mention anything about how he concluded to this statement. But he was aware of the work of other scientists before the publication of the Principia. This is attested by a letter to Robert Hook on February 15, 1676, where Newton

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had written: "If I have seen further it is by standing on the shoulders of Giants." Undoubtedly, Galileo Galilei (1564–1642) was one of these giants. His book on "Two New Sciences" was published in 1638 [3]. Therefore, his experiments with the inclined plane and in general his work on motion were known since then. The question that is posed is: Could Galileo's experimental data from the inclined plane be used to derive Newton's law of motion? As we will show, the answer is yes. This is the subject of the present work.

Galileo's data cannot be directly used because the documents on which he recorded careful measurements obtained experimentally are few in number and very sparing in words. The correct analysis of the experimental measurements is not possible without Galileo's original data behind that set of numbers [4]. But since Galileo could measure distances along the inclined plane and the corresponding times of the rolling ball, we can produce as many Galileo's data as required to derive Newton's law of motion using the system identification procedure we describe below. The implementation of this procedure uses the concepts of the derivative and differential equation. Both these concepts were known before the publication of the law of motion by Newton [5]. Thus, we are allowed to say that the employed identification procedure could have been used to derive Newton's law of motion using Galileo's experiments with the inclined plane before this law was published by Newton. First, the system identification method is presented, and subsequently, it is employed to derive the law of motion for (a) constant mass, (b) variable mass, and (c) the rocket equation.

2 The system identification method

This method has been presented for boundary value problems by Katsikadelis [6]. Here, it is applied to initial value problems.

Consider the differential equation

$$T(s) = f, \tag{2}$$

where $s = s(t)$ and $f = f(t)$. The identification problem is stated as follows:

> Given the $i = 1, 2, \ldots, n$ sets $s_k^{(i)} = s^{(i)}(t_k)$ (input) and $f_k^{(i)} = f(t_k)$ (output) $k = 1, 2, \ldots, N$ produced by mapping (2), determine the differential operator T.

According to [6], this can be achieved by approximating the operator $T(s)$ by a power series of s and its derivatives, e.g., if it is assumed that $T(s)$ is of the second order, it includes only $s, \dot{s}, \ddot{s}$ and we may write

$$T(s) = \sum_{i,j,k=0}^{2} b_{ijk} s^i \dot{s}^j \ddot{s}^k = b_{000} + b_{100}s + b_{010}\dot{s} + b_{001}\ddot{s}$$
$$+ b_{200}s^2 + b_{110}s\dot{s} + b_{101}s\ddot{s} + b_{020}\dot{s}^2 + b_{011}\dot{s}\ddot{s} + b_{002}\ddot{s}^2, \tag{3}$$

where $b_{ijk} = b_{ijk}(t)$ are coefficients to be determined. In what follows, we present this method by assuming that the polynomial is of the (a) first order and (b) second order.

3 Constant mass

3.1 $T(s)$ is approximated by a first-order polynomial

Equation (2) is written as

$$T(s) = b_0 + b_1 s + b_2 \dot{s} + b_3 \ddot{s} = f. \tag{4}$$

The coefficients b_0, b_1, b_2, b_3 are assumed to be different at different time instants, i.e., $b_{ik} = b_i(t_k), k = 1, 2, \ldots, N$; therefore, $4 \times N$ coefficients should be determined.

In this case, we need four experiments. These yield four sets $s_k^{(i)}, i = 1, 2, 3, 4$ at the $k = 1, 2, \ldots, N$ instants. Then, using $s_k^{(i)}$, we evaluate $\dot{s}_k^{(i)}, \ddot{s}_k^{(i)}$ and we may write Eq. (4) as

$$b_{0k} + b_{1k}s_k^{(i)} + b_2\dot{s}_k^{(i)} + b_{3k}\ddot{s}_k^{(i)} = f_k^{(i)}, \quad i = 1, 2, 3, 4,\ k = 1, 2, \ldots, N. \tag{5}$$

Equation (5) represents a system of $4N$ linear algebraic equations for the $4N$ unknowns $b_{0k}, b_{1k}, b_{2k}, b_{3k}, k = 1, 2, \ldots, N$, which can be solved to give the unknown coefficients. For this purpose, Eq. (5) is rearranged as follows:

$$\begin{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} s_1^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_N^{(1)} \end{bmatrix} & \begin{bmatrix} \dot{s}_1^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \dot{s}_N^{(1)} \end{bmatrix} & \begin{bmatrix} \ddot{s}_1^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddot{s}_N^{(1)} \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} s_1^{(2)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_N^{(2)} \end{bmatrix} & \begin{bmatrix} \dot{s}_1^{(2)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \dot{s}_N^{(2)} \end{bmatrix} & \begin{bmatrix} \ddot{s}_1^{(2)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddot{s}_N^{(2)} \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} s_1^{(3)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_N^{(3)} \end{bmatrix} & \begin{bmatrix} \dot{s}_1^{(3)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \dot{s}_N^{(3)} \end{bmatrix} & \begin{bmatrix} \ddot{s}_1^{(3)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddot{s}_N^{(3)} \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} s_1^{(4)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_N^{(4)} \end{bmatrix} & \begin{bmatrix} \dot{s}_1^{(4)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \dot{s}_N^{(4)} \end{bmatrix} & \begin{bmatrix} \ddot{s}_1^{(4)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddot{s}_N^{(4)} \end{bmatrix}
\end{bmatrix}
\begin{Bmatrix} \begin{Bmatrix} b_{01} \\ \vdots \\ b_{0N} \end{Bmatrix} \\ \begin{Bmatrix} b_{11} \\ \vdots \\ b_{1N} \end{Bmatrix} \\ \begin{Bmatrix} b_{21} \\ \vdots \\ b_{2N} \end{Bmatrix} \\ \begin{Bmatrix} b_{31} \\ \vdots \\ b_{3N} \end{Bmatrix} \end{Bmatrix}
=
\begin{Bmatrix} \begin{Bmatrix} f_1^{(1)} \\ \vdots \\ f_N^{(1)} \end{Bmatrix} \\ \begin{Bmatrix} f_1^{(2)} \\ \vdots \\ f_N^{(2)} \end{Bmatrix} \\ \begin{Bmatrix} f_1^{(3)} \\ \vdots \\ f_N^{(3)} \end{Bmatrix} \\ \begin{Bmatrix} f_1^{(4)} \\ \vdots \\ f_N^{(4)} \end{Bmatrix} \end{Bmatrix} \tag{6}$$

or

$$\begin{bmatrix} \mathbf{I} & \mathbf{s}^{(1)} & \dot{\mathbf{s}}^{(1)} & \ddot{\mathbf{s}}^{(1)} \\ \mathbf{I} & \mathbf{s}^{(2)} & \dot{\mathbf{s}}^{(2)} & \ddot{\mathbf{s}}^{(2)} \\ \mathbf{I} & \mathbf{s}^{(3)} & \dot{\mathbf{s}}^{(3)} & \ddot{\mathbf{s}}^{(3)} \\ \mathbf{I} & \mathbf{s}^{(4)} & \dot{\mathbf{s}}^{(4)} & \ddot{\mathbf{s}}^{(4)} \end{bmatrix} \begin{Bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{f}^{(3)} \\ \mathbf{f}^{(4)} \end{Bmatrix}, \tag{7}$$

where $\mathbf{s}^{(i)}, \dot{\mathbf{s}}^{(i)}, \ddot{\mathbf{s}}^{(i)}$ are $N \times N$ diagonal matrices and $\mathbf{b}_i, \mathbf{f}^{(i)}\ N \times 1$ vectors; $\mathbf{I}$ is the unit matrix.

The distances $\mathbf{s}^{(i)}$ can be obtained from Galileo's experiment with the inclined plane (Fig. 1)

3.2 Evaluation of $\dot{\mathbf{s}}^{(i)}$ and $\ddot{\mathbf{s}}^{(i)}$ from Galileo's experimental data

Galileo's experiment gives s_k at time instants t_k. Such data from a virtual experiment with a rolling ball of negligible rotational inertia on an inclined plate with angle $\alpha = 10^{\mathrm{o}}$ are given in Table 1. This set of values is plotted in Fig. 2. Then, the distances at equal times are computed to facilitate the evaluation of the derivatives. This can be done graphically or computationally after interpolating a polynomial through the data. Galileo had found out that "the distance is proportional to the square of time." Therefore, a polynomial of the second order can fit the distance data. The distances at equal times can be used to compute numerically the first and second derivatives using equidistant finite differences, namely

$$\dot{s}_k = \frac{s_{k+1} - s_{k-1}}{2\Delta t}, \quad \ddot{s}_k = \frac{s_{k+1} - 2s_k + s_{k-1}}{\Delta t^2}. \tag{8}$$

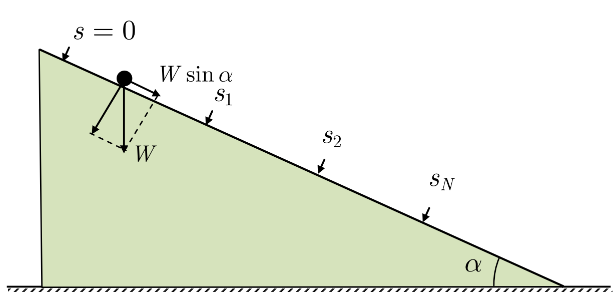

Fig. 1 Schematic diagram of the experiment with the inclined plane

Table 1 Galileo's Data from virtual experiment

Distance s	0.5000	1.0000	1.500	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
Time t	0.7662	1.0835	1.3271	1.5324	1.7132	1.8767	2.0271	2.1671	2.2985	2.4229

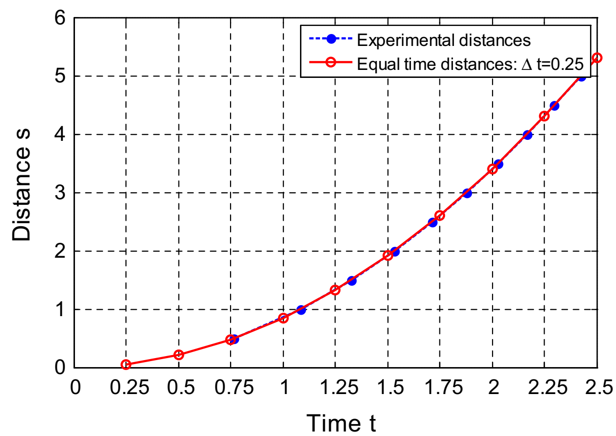

Fig. 2 Distance versus time for Galileo's data

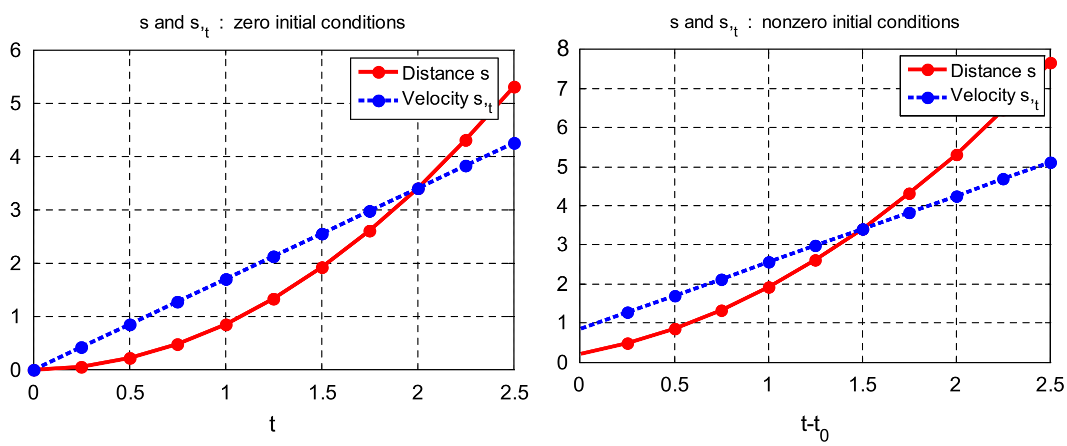

Fig. 3 Galileo's data with zero (**a**) and nonzero (**b**) initial conditions

For the accuracy of the results, central differences are recommended.

This procedure is used to obtain $s_k^{(i)}$, $\dot{s}_k^{(i)}$, $\ddot{s}_k^{(i)}$ for the four experiments corresponding to four different angles α_i of the inclined plane. The coefficient matrix in Eq. (7) is singular, unless the initial conditions s_0, $\dot{s}_0$ are different for each experiment. Since Galileo's experiment on the inclined plane does not allow giving an initial velocity, we can achieve it by shifting origin of time by t_0 in the plot of Fig. 2. This is shown in Fig. 3 for $t_0 = 0.5$.

On the basis of the described procedure, the computed coefficients resulting from the solution of Eq. (7) are given in Table 2. Apparently, it is $b_0 = b_1 = b_2 = 0$ and $b_3 = \text{constant} \neq 0$ as anticipated. This allows to write Eq. (4) as

Table 2 Results for first-order polynomial. Constant mass

$T(s) = b_0 + b_1 s + b_2 \dot{s} + b_3 \ddot{s} = W \sin a$

$N = 10$ $W = 50$				$N = 15$ $W = 30$			
b_0	b_1	b_2	b_3	b_0	b_1	b_2	b_3
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	3.0581
				0.0000	0.0000	0.0000	3.0581
				0.0000	0.0000	0.0000	3.0581
				0.0000	0.0000	0.0000	3.0581
				0.0000	0.0000	0.0000	3.0581
				0.0000	0.0000	0.0000	3.0581
$W/b3 = 9.81$				$W/b3 = 9.81$			

$b_3 \ddot{s} = W \sin a \Rightarrow m\ddot{s} = f(t) \Rightarrow \frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t)$
$b_3 = \text{mass}, \; f(t) = W \sin a = \text{force}$

$$b_3 \ddot{s} = f \tag{9}$$

or

$$\frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t), \quad m = b_3, \tag{10}$$

which yields Newton's law of motion for constant mass. Note that it is always $W/b_3 = 9.81$, that is, the acceleration of gravity.

3.3 $T(s)$ is approximated by a second-order polynomial

Let Eq. (2) have the form

$$T(s) = b_0 + b_1 s + b_2 \dot{s} + b_3 \ddot{s} + b_4 \dot{s}^2 + b_5 \dot{s}\ddot{s} + b_6 \ddot{s}^2 = f(t), \quad f(t) = W \sin a. \tag{11}$$

The coefficients $b_0, b_1, b_2, b_3, b_4, b_5, b_6$ are assumed to be different at different time instants, i.e., $b_{ik} = b_i(t_k)$, $k = 1, 2, \ldots, N$; therefore, $7 \times N$ coefficients should be determined.

In this case, we need seven experiments and Eq. (7) becomes

$$\begin{bmatrix} \mathbf{I} & \mathbf{s}^{(1)} & \dot{\mathbf{s}}^{(1)} & \ddot{\mathbf{s}}^{(1)} & \dot{\mathbf{s}}^{(1)}\dot{\mathbf{s}}^{(1)} & \dot{\mathbf{s}}^{(1)}\ddot{\mathbf{s}}^{(1)} & \ddot{\mathbf{s}}^{(1)}\ddot{\mathbf{s}}^{(1)} \\ \mathbf{I} & \mathbf{s}^{(2)} & \dot{\mathbf{s}}^{(2)} & \ddot{\mathbf{s}}^{(2)} & \dot{\mathbf{s}}^{(2)}\dot{\mathbf{s}}^{(2)} & \dot{\mathbf{s}}^{(2)}\ddot{\mathbf{s}}^{(2)} & \ddot{\mathbf{s}}^{(2)}\ddot{\mathbf{s}}^{(2)} \\ \mathbf{I} & \mathbf{s}^{(3)} & \dot{\mathbf{s}}^{(3)} & \ddot{\mathbf{s}}^{(3)} & \dot{\mathbf{s}}^{(3)}\dot{\mathbf{s}}^{(3)} & \dot{\mathbf{s}}^{(3)}\ddot{\mathbf{s}}^{(3)} & \ddot{\mathbf{s}}^{(3)}\ddot{\mathbf{s}}^{(3)} \\ \mathbf{I} & \mathbf{s}^{(4)} & \dot{\mathbf{s}}^{(4)} & \ddot{\mathbf{s}}^{(4)} & \dot{\mathbf{s}}^{(4)}\dot{\mathbf{s}}^{(4)} & \dot{\mathbf{s}}^{(4)}\ddot{\mathbf{s}}^{(4)} & \ddot{\mathbf{s}}^{(4)}\ddot{\mathbf{s}}^{(4)} \\ \mathbf{I} & \mathbf{s}^{(5)} & \dot{\mathbf{s}}^{(5)} & \ddot{\mathbf{s}}^{(5)} & \dot{\mathbf{s}}^{(5)}\dot{\mathbf{s}}^{(5)} & \dot{\mathbf{s}}^{(5)}\ddot{\mathbf{s}}^{(5)} & \ddot{\mathbf{s}}^{(5)}\ddot{\mathbf{s}}^{(5)} \\ \mathbf{I} & \mathbf{s}^{(6)} & \dot{\mathbf{s}}^{(6)} & \ddot{\mathbf{s}}^{(6)} & \dot{\mathbf{s}}^{(6)}\dot{\mathbf{s}}^{(6)} & \dot{\mathbf{s}}^{(6)}\ddot{\mathbf{s}}^{(6)} & \ddot{\mathbf{s}}^{(6)}\ddot{\mathbf{s}}^{(6)} \\ \mathbf{I} & \mathbf{s}^{(7)} & \dot{\mathbf{s}}^{(7)} & \ddot{\mathbf{s}}^{(7)} & \dot{\mathbf{s}}^{(7)}\dot{\mathbf{s}}^{(7)} & \dot{\mathbf{s}}^{(7)}\ddot{\mathbf{s}}^{(7)} & \ddot{\mathbf{s}}^{(7)}\ddot{\mathbf{s}}^{(7)} \end{bmatrix} \begin{Bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \mathbf{b}_5 \\ \mathbf{b}_6 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{f}^{(3)} \\ \mathbf{f}^{(4)} \\ \mathbf{f}^{(5)} \\ \mathbf{f}^{(6)} \\ \mathbf{f}^{(7)} \end{Bmatrix}. \tag{12}$$

Results for the computed coefficients when the operator is approximated by a second-order polynomial are given in Table 3. Galileo's data from seven experiments have been employed. Obviously, Eq. (11) takes again the form of Eq. (10).

The results in both cases have been obtained using a relatively large number of times for each experiment: $N = 10$ and $N = 15$ for first-order polynomial and $N = 7$ and $N = 10$ for second-order polynomial. These

Table 3 Results for second-order polynomial. Constant mass

$T(s) = b_0 + b_1 s + b_2\dot{s} + b_3\ddot{s} + b_4\dot{s}^2 + b_5\dot{s}\ddot{s} + b_6\ddot{s}^2 = W \sin a$

$N = 7$ $W = 50$							$N = 10$ $W = 30$						
b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_0	b_1	b_2	b_3	b_4	b_5	b_6
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.0968	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
							0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
							0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
							0.0000	0.0000	0.0000	3.0581	0.0000	0.0000	0.0000
$W/b3 = 9.81$							$W/b3 = 9.81$						

$b_3\ddot{s} = W \sin a \Rightarrow m\ddot{s} = f(t) \Rightarrow \frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t)$
$b_3 = \text{mass},\ f(t) = W \sin a = \text{force}$

Table 4 Results for different values of N. First-order polynomial. Constant mass

$T(s) = b_0 + b_1 s + b_2\dot{s} + b_3\ddot{s} = W \sin a,\quad W = 10, a = 10°, 20°, 30°, 40°$

N	t	b_0	b_1	b_2	b_3
1	1.0	0.0000	0.0000	−0.0000	1.0194
2	1.0	0.0000	0.0000	−0.0000	1.0194
	3.0	0.0000	0.0000	−0.0000	1.0194
3	1.0	0.0000	0.0000	−0.0000	1.0194
	3.0	0.0000	0.0000	−0.0000	1.0194
	5.0	0.0000	0.0000	−0.0000	1.0194
4	1.0	0.0000	0.0000	−0.0000	1.0194
	3.0	0.0000	0.0000	−0.0000	1.0194
	5.0	0.0000	0.0000	−0.0000	1.0194
	7.0	0.0000	0.0000	−0.0000	1.0194

numbers of time instances lead to large systems of linear equations, which actually can be solved using a computer program. In this analysis, the MATLAB code has been employed to derive the numerical results. However, it is not necessary to use large matrices $s_k^{(i)}, \dot{s}_k^{(i)}, \ddot{s}_k^{(i)}, k = 1, 2, \ldots, N$. The same results are obtained for $N = 1$ or $N = 2$, which yield very small systems that can be solved without the use of a machine. Table 4 gives results for different values of N. Of course, at least two values are required to show that the coefficients are constant.

4 Variable mass

So far, the data have been obtained for a system with constant mass (small ball with negligible rotational inertia). For a variable mass system, the experiment should be performed with a system having mass ablation with increasing time. A schematic setup for such an experiment is shown in Fig. 4. It may consist of a closed tank with two side openings (normal to the motion), which slides without friction on the inclined plane. The tank is filled with water of mass $m_0 = W_0/g$ at $t = 0$. The law of the water flow through the openings can be established experimentally. Since we do not perform actual experiments, we assume that the water flow obeys the exponential law and we acquire the numerical data from the analytical solution of the equation of motion, namely

$$\frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t), \quad s(0) = 0, \quad \dot{s}(0) = 0, \tag{13}$$

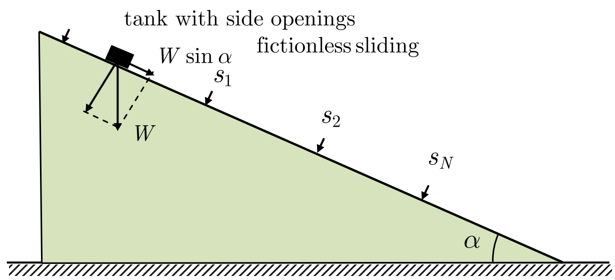

Fig. 4 Schematic diagram of the inclined plane experiment with variable mass

Table 5 Results for variable mass. First- and second-order polynomial approximation of $T(s)$

$N = 10$, $W_0 = 20$, $c = 0.2$										
$T(s) = b_0 + b_1 s + b_2\dot{s} + b_3\ddot{s}$				$T(s) = b_0 + b_1 s + b_2\dot{s} + b_3\ddot{s} + b_4\dot{s}^2 + b_5\dot{s}\ddot{s} + b_6\ddot{s}^2$						
b_0	b_1	b_2	b_3	b_0	b_1	b_2	b_3	b_4	b_5	b_6
0.0000	0.0000	−0.3997	1.9984	0.0000	0.0000	−0.3997	1.9984	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3840	1.9200	0.0000	0.0000	−0.3840	1.9200	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3689	1.8447	0.0000	0.0000	−0.3689	1.8447	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3545	1.7724	0.0000	0.0000	−0.3545	1.7724	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3406	1.7029	0.0000	0.0000	−0.3406	1.7029	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3272	1.6361	0.0000	0.0000	−0.3272	1.6361	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3144	1.5720	0.0000	0.0000	−0.3144	1.5720	0.0000	0.0000	0.0000
0.0000	0.0000	−0.3021	1.5103	0.0000	0.0000	−0.3021	1.5103	0.0000	0.0000	0.0000
0.0000	0.0000	−0.2902	1.4511	0.0000	0.0000	−0.2902	1.4511	0.0000	0.0000	0.0000
0.0000	0.0000	−0.2788	1.3942	0.0000	0.0000	−0.2788	1.3942	0.0000	0.0000	0.0000
$W/b3 = 9.81$				$W/b3 = 9.81$						
$b_2\dot{s} + b_3\ddot{s} = W\sin a$, $\dot{b}_3 = b_2 \Rightarrow \frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t)$, $b_3 =$ mass, $f(t) = W\sin a =$ force										

where

$$m = \frac{W_0}{g}e^{-ct}, \quad f(t) = W\sin a, \quad W = W_0 e^{-ct}, \tag{14a,b}$$

with c being a parameter.

The initial value problem (13) admits an exact solution

$$s(t) = \frac{g\sin a}{c}\left(-t + \frac{e^{ct}}{c} - \frac{1}{c}\right). \tag{15}$$

This solution is employed to obtain Galileo's experimental data for the system identification procedure. The computed coefficients obtained with $N = 10$ for linear approximation (four experiments) and nonlinear approximation (seven experiments) of the operator $T(s)$ are given in Table 5. As it was anticipated, it is $\mathrm{d}b_3/\mathrm{d}t = b_2$, (see Tables 5, 6), which produces Newton's law of motion when the mass is variable, if $b_3 = m$ is set, i.e.,

$$b_2\dot{s} + b_3\ddot{s} = \frac{\mathrm{d}(b_3\dot{s})}{\mathrm{d}t} = f(t) \quad \text{or} \quad \frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t) \tag{16}$$

5 Mass ejection: The rocket equation

In this case, the differential equation describes the motion of a system, which ejects (or gains) mass during the motion, i.e., it is a system with variable mass. The difference, however, from the variable mass system treated in Sect. 4 is that the velocity of the ejecting (or entering) mass acts in the direction of motion and influences it. Newton's second law of motion cannot directly be applied to such a system [7]. Instead, the time dependence

Table 6 Mass and its derivative. $N = 10$, $W_0 = 20$, $c = 0.2$

Mass		Derivative of mass	
$m = (W/g)\,e^{-ct}$	b_3	$\dot{m} = -(W/g)\,ce^{-ct}$	b_2
1.9984	1.9984	−0.3997	−0.3997
1.9200	1.9200	−0.3840	−0.3840
1.8447	1.8447	−0.3689	−0.3689
1.7724	1.7724	−0.3545	−0.3545
1.7029	1.7029	−0.3406	−0.3406
1.6361	1.6361	−0.3272	−0.3272
1.5720	1.5720	−0.3144	−0.3144
1.5103	1.5103	−0.3021	−0.3021
1.4511	1.4511	−0.2902	−0.2902
1.3942	1.3942	−0.2788	−0.2788

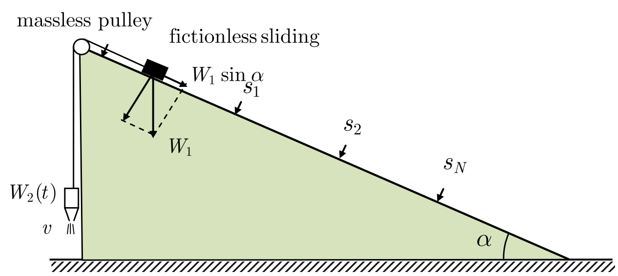

Fig. 5 Schematic diagram of the inclined plane experiment with mass ejection

of the mass m can be considered by rearranging Newton's second law and adding a term to account for the momentum carried by mass entering or leaving the system. The general equation of variable mass motion, known also as the *rocket equation*, is written as [8]

$$m\ddot{s} - \mathrm{v}_r\dot{m} = f(t), \tag{17}$$

where $\mathrm{v}_r = v - \dot{s}$ is the relative velocity between the velocity v of the entering or leaving mass and the velocity $\dot{s}$ of the system. Rearranging Eq. (17), we may write

$$m\ddot{s} + \dot{m}\dot{s} - \dot{m}v = f(t). \tag{18}$$

Apparently, if $v = 0$, Eq. (18) becomes identical to Eq. (13). We observe that the term $\dot{m}v$ gives the system an additional external force, positive or negative depending on the direction of v.

The identification procedure to derive Eq. (18) can be applied by performing experiments with a system of variable mass with mass ablation (or accretion) during the motion. A schematic setup for such an experiment is shown in Fig. 5. It consists of a body with constant mass $m_1 = W_1/g$ sliding without friction on the inclined plane and a bucket containing water with an opening at the bottom, which allows the water flowing. The string is assumed tensionless. The bucket is filled with water of mass $m_0 = W_0/g$ at $t = 0$. The law of the water flow through the opening can be established experimentally. As previously, we assume that the water flow obeys the exponential law, i.e., $W_2(t) = W_0e^{-ct}$. Thus, we have

$$m = [W_0\exp(-ct) + W_1]/g, \tag{19a}$$

$$f(t) = [W_1 \sin a - W_0\exp(-ct)] > 0, \tag{19b}$$

$$v = -de^{-ct/2}, \quad d = \sqrt{2W_0/A}, \quad A = \text{the cross section of the bucket.} \tag{19c}$$

We acquire the data from virtual experiments by solving the equation of motion, Eq. (18). The numerical solution is preferred [9], because the expression of the obtained analytical solution is very lengthy, complicated and includes special functions which are not available in the computer mathematical libraries, hence difficult to evaluate. The employed data should ensure $f(t) = [W_1 \sin a - W_0\exp(-ct)] > 0$. The computed coefficients for second-order polynomial approximation of $T(s)$ are given in Table 7 for different values of N. It is observed that the coefficient b_0 does not vanish. It represents the additional force $b_0 = v\dot{m}$ due to the ejection of the mass. The results in Table 8 illustrate this.

Table 7 Results for the rocket equation. Second-order polynomial approximation of $T(s)$

$W_0 = 10$, $W1 = 20$, $c = 0.1$, $d = 1$

$T(s) = b_0 + b_1 s + b_2\dot{s} + b_3\ddot{s} + b_4\dot{s}^2 + b_5\dot{s}\ddot{s} + b_6\ddot{s}^2$

N	t	b_0	b_1	b_2	b_3	b_4	b_5	b_6
1	0.5	−0.0946	0.0000	−0.0970	3.0084	−0.0000	0.0000	−0.0000
2	0.5	−0.0946	0.0000	−0.0970	3.0084	−0.0000	0.0000	−0.0000
	1.0	−0.0877	0.0000	−0.0922	2.9611	−0.0000	0.0000	−0.0000
3	0.5	−0.0946	0.0000	−0.0970	3.0084	−0.0000	0.0000	−0.0000
	1.0	−0.0877	0.0000	−0.0922	2.9611	−0.0000	0.0000	−0.0000
	1.5	−0.0814	0.0000	−0.0877	2.9161	−0.0000	0.0000	−0.0000
4	0.5	−0.0946	0.0000	−0.0970	3.0084	−0.0000	0.0000	−0.0000
	1.0	−0.0877	0.0000	−0.0922	2.9611	−0.0000	0.0000	−0.0000
	1.5	−0.0814	0.0000	−0.0877	2.9161	−0.0000	0.0000	−0.0000
	2.0	−0.0755	0.0000	−0.0835	2.8733	−0.0000	0.0000	−0.0000
5	0.5	−0.0946	0.0000	−0.0970	3.0084	−0.0000	0.0000	−0.0000
	1.0	−0.0877	0.0000	−0.0922	2.9611	−0.0000	0.0000	−0.0000
	1.5	−0.0814	0.0000	−0.0877	2.9161	−0.0000	0.0000	−0.0000
	2.0	−0.0755	0.0000	−0.0835	2.8733	−0.0000	0.0000	−0.0000
	2.5	−0.0701	0.0000	−0.0794	2.8326	−0.0000	0.0000	−0.0000

$b_0 + b_2\dot{s} + b_3\ddot{s} = W_1 \sin a - W_2$, $\dot{b}_3 = b_2 \Rightarrow \frac{\mathrm{d}(m\dot{s})}{\mathrm{d}t} = f(t)$,
$b_3 = \text{mass}$, $b_0 = v\dot{m}$, $f(t) = W_1 \sin a - W_2 - b_0 = \text{total external force}$

Table 8 Results for the rocket equation. Second-order polynomial approximation of $T(s)$

$N = 5$, $W_0 = 10$, $W1 = 20$, $c = 0.1$, $d = 1$

$T(s) = b_0 + b_1 s + b_2\dot{s} + b_3\ddot{s} + b_4\dot{s}^2 + b_5\dot{s}\ddot{s} + b_6\ddot{s}^2$

t	m	b_3	$\dot{m}$	b_2	b_0	$-v\dot{m}$
0.5	3.0084	3.0084	−0.0970	−0.0970	−0.0946	−0.0946
1.0	2.9611	2.9611	−0.0922	−0.0922	−0.0877	−0.0877
1.5	2.9161	2.9161	−0.0877	−0.0877	−0.0814	−0.0814
2.0	2.8733	2.8733	−0.0835	−0.0835	−0.0755	−0.0755
2.5	2.8326	2.8326	−0.0794	−0.0794	−0.0701	−0.0701

$b_3 = m$, $b_2 = \dot{m}$, $b_0 = -v\dot{m}$,

6 Conclusions

A simple system identification method is presented, which is employed to derive Newton's law of motion using only Galileo's experimental data from the inclined plane. The method is based on the concepts of the derivative and the differential equation together with the capability to solve systems of linear algebraic equations. All these mathematical tools were known before 1686, when Newton published his law of motion as an axiom. Therefore, we are allowed saying that equation of motion could have been derived using Galileo's experiments before Newton had published his second law.

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