2. Infine Series.

An infinite series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 \dots$$

converge to the sum S provided the partial sums has the limit S:

$$\lim_{N \to \infty} \left(\sum_{n=1}^{N} a_n \right) = S$$

$$\sum_{n=1}^{\infty} a_n = S$$

The series converges absolutely if:

$$\sum_{n=1}^{\infty} |a_n| \quad \text{converges.}$$

Absolute convergence $\Rightarrow$ convergence
$\not\Leftarrow$

eg: $1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges ($\ln 2$)

but it doesn't converge absolutely

$1 + \frac{1}{2} + \frac{1}{3} \dots$ does not converge

Convergence critiria:

- Compare the series with a known convergent or divergent series.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

converges for $|x| < 1$

diverges for $|x| > 1$

ratio test:

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ converges (absolutely)

$\lim \left| \frac{a_{n+1}}{a_n} \right| > 1$ diverges

- Compare with an infinite integral.

$$f(1) + f(2) + f(3) + \cdots$$

Converges or diverges with the infinite integral:

$$\int^{\infty} f(x)\, dx$$

for $f(x)$ monotonically decreasing.

e.g. Riemann zeta function.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

(converges or diverges)?

$$\frac{a_{n+1}}{a_n} = \left(\frac{1}{n+1}\right)^s \frac{1}{\left(\frac{1}{n}\right)^s} = \left(\frac{n}{n+1}\right)^s = \left(1 + \frac{1}{n}\right)^{-s} \underset{n\to\infty}{\sim} 1 - \frac{s}{n}$$

$$\frac{a_{n+1}}{a_n} \to 1 \quad \text{for} \quad n \to \infty$$

ratio test fails.

integral test: $\int^\infty \frac{dx}{x^s} = \frac{-1}{s-1} \frac{1}{x^{s-1}}$ finite for $s > 1$

$\zeta(s)$ converges for $s > 1$

improved ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| \to 1 - \frac{s}{n}$$

series converges for $s > 1$

further improvement of ratio test:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^s} = \frac{1}{2(\ln 2)^s} + \frac{1}{3(\ln 3)^s} + \dots$$

comparison with integral:

$$\int \frac{dx}{x(\ln x)^s} = -\frac{1}{s-1} \frac{1}{(\ln x)^{s-1}}$$

finite for $s > 1$

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \left[\frac{\ln n}{\ln(n+1)} \right]^s =$$

$$= \left(1 - \frac{1}{n} + \dots\right) \left[\frac{\ln n}{\ln n + \ln(1+\frac{1}{n})} \right]^s =$$

$$= \left(1 - \frac{1}{n} + \dots\right) \left[1 + \frac{1}{n \ln n} + \dots \right]^{-s}$$

$$= 1 - \frac{1}{n} - \frac{s}{n \ln n}$$

converges for $\underline{s > 1}$.

Hypergeometric:

$$F(a,b,c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(a+n)(b+n)}{(c+n)(n+1)} x = \frac{(1+a/n)(1+b/n)}{(1+c/n)(1+1/n)} x$$

2.2. Familiar Series.

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + \dots = \sum_{m=0}^{\infty} \frac{n!}{(n-m)!}\frac{x^m}{m!}$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

Bernoulli numbers:

$$\frac{x}{e^x - 1} = c_0 + c_1 x + \dots = \sum c_n x^n$$

$$c_n = \frac{B_n}{n!} \rightarrow \text{Bernoulli}$$

$B_0 = 1$ $\qquad B_1 = -\frac{1}{2}$

$B_2 = \frac{1}{6}$ $\qquad B_3 = B_5 = \dots 0$

$B_4 = -\frac{1}{30}$

$B_6 = \frac{1}{\dots}$

eg: $$\operatorname{ctn} x = \frac{\cos x}{\sin x}$$

$ix = y/2$

$$\operatorname{ctn} x = i \frac{e^{y/2} + e^{-y/2}}{e^{y/2} - e^{-y/2}} = i\left(1 + \frac{2}{e^y - 1}\right)$$

$$= \frac{2i}{y}\left(\frac{y}{2} + \frac{y}{e^y - 1}\right) = \frac{2i}{y} \sum_{n \in \text{even}} \frac{B_n y^n}{n!}$$

$$\operatorname{ctn} x = \frac{1}{x} \sum_{n \text{ even}} (-1)^{n/2} \frac{B_n (2x)^n}{n!}$$

$$\tan x = \operatorname{ctn} x - 2 \operatorname{ctn} 2x$$

we may find the expansion of $\tan x$.

2-3. Transformation of series.

- Differentiation & integration.

e.g: $f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$$\int^x f(x)\,dx = x + x^2 + x^3 + \cdots = x(1 + x^2 + x^3 \cdots) = \frac{x}{1-x}$$

$$\Rightarrow \quad f(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}$$

$$f(x) = \frac{1}{1\cdot 2} + \frac{x}{2\cdot 3} + \frac{x^2}{3\cdot 4} + \frac{x^3}{4\cdot 5} + \cdots$$

$$x^2 f(x) = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^5}{3\cdot 4} + \cdots$$

$$(x^2 f)'' = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$f = \int^x \int^y \frac{1}{1-t}\,dt = \frac{1}{x} + \frac{1-x}{x^2}\ln(1-x)$$

Constant of integration $f(x)$ at special values $(x=0)$

$$S = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} \cdots$$

Define

$$f(x) = \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \cdots$$

$$S = f(1)$$

$$f' = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots = x e^x$$

$$f(x) = \int^x x e^x dx = x e^x - e^x + 1$$

$f(0) = 0$

$$S = f(1) = 1$$

$$S = 1 + m + \frac{m(m-1)}{2!} + \cdots = \sum_n \frac{m!}{n!(n-m)!}$$

$$f(x) = \sum_n \frac{m!}{n!(m-n)!} x^n \qquad S = f(1)$$

$$f(x) = (1+x)^m \qquad S = 2^m$$